

Height-Diameter Equations

7

Tree heights are required for the calculation of volume, as explained in Chapter 3. When tree heights are not measured for each tree (e.g. – stump cruises) height-diameter equations, or curves, are derived from the sample trees.

Heights are assigned by substituting the DBH into the chosen equation. Each equation has a minimum value DBH_{min} , below which the equation predicts negative values. Each function also has an associated DBH_{max} . Heights cannot be calculated for DBHs greater than DBH_{max} or lower than DBH_{min} . Where a DBH does not fall into the acceptable range for the equation, the minimum or maximum DBH would be substituted, and a message would be indicated for that tree in report 004a (see Figure 7.2).

There are 6 possible equations available:

| | |
|-----------------------|--------------------------------------|
| Parabola | $H = a + bD + cD^2$ |
| Conditioned Parabola | $H = 1.3 + bD + cD^2$ |
| Hyperbola | $H = a + b/D + cD$ |
| Conditioned Hyperbola | $H = 1.3 - bD/(D+1) + cD$ |
| Weibull | $H = a[1 - \text{EXP}(-bD^c)]$ |
| Conditioned Weibull | $H = 1.3 + a[1 - \text{EXP}(-bD^c)]$ |

Where:

H = Tree height in metres, to the nearest tenth.

D = Diameter at 1.3 m outside bark in centimeters, to the nearest tenth.

a, b, c = Regression coefficients.

EXP(x) = e^x , where e is the base of the natural logarithm function.

Use the following method to choose the suggested curve for assigning heights:

1. Reject any curves that compute a negative height for a DBH of 10 cm or lower.
2. If the b and c coefficients are positive for the Parabola, the Weibull function must not be used.
3. Of the remaining curves, choose the curve with the lowest standard error estimate (S.E.E.) based on volume. **If the curve with the lowest SEE volume exceeds 3% over the range of DBHs, then another curve may be used in the compilation.**

Regression equations are calculated for each species within a Timber Type. In order to produce regression coefficients, a minimum of 20 sample trees must be used. For minor species (less than twenty percent of the unauthorized timber harvest area gross volume) ten trees per curve is acceptable. If there are not enough sample trees for a species in a Timber Type, then samples may be grouped together for different species and types using card type E. Where more than one species contributes to a height curve, appropriate, to each species volume constants, as per Appendix 15 are to be applied when calculating the S.E.E. volume. Refer to **Chapter 6** of the [Cruising Manual](#) for a description of the methods of grouping sample trees.

A height curve equation applies heights for all of the Species and Types listed in columns 15-80 of card type E.

If the column 14 of card type E is coded '1', then the sample trees used to derive the height curve are restricted to the Species and Type coded in the Species Used and the Type Used fields.

If column 14 is blank or 0, then the sample trees used to derive the height curve would be all of the Species and Types in columns 15-80 of card type E. Refer to **Chapter 6** of the [Cruising Manual](#) for example card type E codes.

A good practice is to first plot the sample trees diameter vs. height. This provides a useful picture of the height-diameter relationship for the species and types that the equation generates heights for. It will also indicate outliers if they exist.

Next, find the regression coefficients for all six curves. To determine the S.E.E. volume, use the method described in Section 7.3. Choose the curve that yields the lowest S.E.E. for volume as the suggested curve that does not generate negative heights at 10 cm DBH or has a maximum/minimum DBH range insufficient for the card Type 2 requirements.

7.1 Regression Coefficients

In order to determine the regression coefficients, there are many statistical packages available which support the above equations. For the parabola and hyperbola equations, fit a linear least-squares regression with an intercept. For the Weibull functions, fit a non-linear least squares regression. For the conditioned forms of each equation type, change the dependent variable to $H - 1.3$, rather than H and fit the equation without an intercept.

A method for determining coefficients for the parabola and hyperbola is included for each equation listed below.

In the following examples, N = the number of sample trees used.

The Σ without upper and lower bound parameters in the regression coefficient formulas is a summation over all N of the sample trees.

7.1.1 Parabola

$$A = \Sigma HD^2 - \frac{\Sigma H \Sigma D^2}{N}$$

$$B = \Sigma D^2 - \frac{(\Sigma D)^2}{N}$$

$$C = \Sigma D^3 - \frac{\Sigma D^2 \Sigma D}{N}$$

$$D = \Sigma HD - \frac{\Sigma H \Sigma D}{N}$$

$$E = \Sigma D^4 - \frac{(\Sigma D^2)^2}{N}$$

$$F = \Sigma D$$

$$G = \Sigma H$$

$$H = \Sigma D^2$$

The parabola regression coefficients are calculated as follows:

$$c = \frac{AB - CD}{BE - CC}$$

$$b = \frac{D - cC}{B}$$

$$a = \frac{(G - bF - cH)}{N}$$

7.1.2 Conditioned Parabola

$$A = \sum(H - 1.3)D^2$$

$$B = \sum D^2$$

$$C = \sum D^3$$

$$D = \sum(H - 1.3)D$$

$$E = \sum D^4$$

The conditioned parabola regression coefficients are calculated as follows:

$$c = \frac{AB - CD}{BE - CC}$$

$$b = \frac{D - cC}{B}$$

$$a = 1.3$$

7.1.3 Hyperbola

$$A = \sum HD - \frac{\sum H \sum D}{N}$$

$$B = \sum \frac{1}{D^2} - \frac{\left(\sum 1/D\right)^2}{N}$$

$$C = N - \frac{\sum 1/D \sum D}{N}$$

$$D = \sum \frac{H}{D} - \frac{\sum H \sum 1/D}{N}$$

$$E = \sum D^2 - \frac{(\sum D)^2}{N}$$

$$F = \sum \frac{1}{D}$$

$$G = \sum H$$

$$H = \sum D$$

The hyperbola regression coefficients are calculated as follows:

$$c = \frac{AB - CD}{BE - CC}$$

$$b = \frac{D - cC}{B}$$

$$a = \frac{(G - bF - cH)}{N}$$

7.1.4 Conditioned Hyperbola

$$A = \sum (H - 1.3)D$$

$$B = \sum \frac{(-D)^2}{(D + 1)^2}$$

$$C = \sum \frac{-(D^2)}{(D + 1)}$$

$$D = \sum \frac{-D(H - 1.3)}{(D + 1)}$$

$$E = \sum D^2$$

The conditioned hyperbola regression coefficients are calculated as follows:

$$c = \frac{AB - CD}{BE - CC}$$

$$b = \frac{D - cC}{B}$$

$$a = 1.3$$

7.1.5 Weibull and Conditioned Weibull

The Weibull coefficients are not determined by matrix manipulation as the other two curves are. The calculations will not be provided here. Instead, use T. Kozak's subroutine - April, 1993. For the un-conditioned Weibull, the dependant variable would be the set of sample tree heights (H) and the independent variable would be the set of sample tree diameters (D). For the conditioned Weibull, the dependent variable would be the set of sample tree heights less 1.3 m (i.e., H-1.3).

When coefficients are determined through iteration the maximum number of iterations is 1 000.

7.2 Minimum and Maximum DBH

Heights can only be calculated for trees when their DBH is in the specified range for the height-diameter curve. When a tree falls outside of the DBH range, use either the height calculated for DBH_{max} or DBH_{min} for the selected curve.

7.2.1 Parabola and Conditioned Parabola

7.2.1.1 Minimum DBH

For the conditioned and un-conditioned parabola:

DBH_{min} = the root that is closest to 0:

$$DBH_{min} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

The root closest to zero of the two roots is selected.

When $b^2 - 4ac < 0$, then set $DBH_{min} = 0$.

Example 7.1

$$a = 1.3$$

$$b = 0.62079155$$

$$c = -0.00307971$$

$$\text{Root}_1 = -2.0728 \quad \leftarrow DBH_{min}$$

$$\text{Root}_2 = 203.647$$

7.2.1.2 Maximum DBH

This method can solve either the minimum or maximum DBH depending on the values of the coefficients. If b is positive and c is negative there is a maximum height at a given DBH. If both b and c are positive there is a minimum height at a given DBH. A maximum DBH_{\max} of 250 cm has been assigned.

$$DBH_{\max} = \text{Root of the derivative of the parabola}$$

$$H = a + bD + cD^2$$

$$H' = b + 2cD$$

Setting derivative equal to zero, gives

$$DBH_{\max} = \frac{-b}{2c}$$

Example 7.2

$$b = 0.62079155$$

$$c = -0.00307971$$

$$DBH_{\max} = 100.7873$$

7.2.2 Hyperbola and Conditioned Hyperbola

7.2.2.1 Minimum DBH

For the hyperbola, DBH_{\min} = the root that is closest to 0 of the following quadratic equation.

$$cD^2 + aD + b = 0$$

$$DBH_{\min} = \frac{-a \pm \sqrt{a^2 - 4cb}}{2c}$$

When $a^2 - 4cb < 0$, then set $DBH_{\min} = 0$.

Example 7.3

$$a = 10.81912465$$

$$b = -9.74355472$$

$$c = 0.23902870$$

$$\text{Root}_1 = 0.8833 \quad \leftarrow \text{DBH}_{\min}$$

$$\text{Root}_2 = -46.1462$$

For the conditioned hyperbola, DBH_{\min} = The root that is closest to 0 of the following quadratic equation

$$cD^2 + (c-b+a)D + a = 0$$

$$\text{DBH}_{\min} = \frac{(b-a-c) \pm \sqrt{(c-b+a)^2 - 4ac}}{2c}$$

When $(c-b+a)^2 - 4cb < 0$, then set $\text{DBH}_{\min} = 0$.

Example 7.4

$$a = 1.3$$

$$b = -9.48876491$$

$$c = 0.23935168$$

$$\text{Root}_1 = -0.1182 \quad \leftarrow \text{DBH}_{\min}$$

$$\text{Root}_2 = -45.9567$$

7.2.2.2 Maximum DBH

If b is negative and c is positive there is no maximum or minimum DBH.

If both b and c are negative then $DBH_{max} = (b/c)^{1/2}$

To estimate DBH_{max} , calculate the height at increasing 0.5 cm intervals for DBH. Pick the diameter that yields the greatest height. The calculated heights will increase with increasing DBH, until the vertex (DBH_{max}) is passed. At that point, the heights will decrease with increasing DBH. Set D_{max} to the 0.5 cm interval before the calculated heights started to decrease. If the vertex is not passed before 250 cm, set DBH_{max} equal to 250 cm. In this case, the reported DBH_{max} is not a true maximum, but rather an arbitrarily assigned maximum.

Example 7.5

| DBH | Height | |
|-------|--------|---------------|
| 170 | 51.4 | |
| 170.5 | 51.5 | |
| 171 | 51.6 | ← DBH_{max} |

Example 7.6

| DBH | Height | |
|-------|--------|---------------|
| 122 | 49.0 | |
| 122.5 | 49.1 | ← DBH_{max} |
| 123 | 49.0 | |

7.2.3 Weibull and Conditioned Weibull

7.2.3.1 Minimum DBH

DBH_{min} for the Weibull function is 0.

7.2.3.2 Maximum DBH

DBH_{max} for the Weibull function is infinity, since a is the maximum height at infinity DBH. A maximum DBH_{max} of 250 cm has been assigned.

7.3 Standard Error of Estimate (S.E.E.) and Bias

Bias is a systematic distortion between the observed and calculated values. Because of the way the formula was created in statistics, if the calculated height were generally higher than the observed heights, the height bias would be negative. Likewise, if the calculated heights were systematically lower than the observed heights, the height bias would be positive.

Standard Error of Estimate is the measure of spread of the observed values around the regression line (estimated values).

7.3.1 Standard Error of Estimate - Height

7.3.1.1 Parabola

$$A = \sum HD^2 - \frac{\sum H \sum D^2}{N}$$

$$D = \sum HD - \frac{\sum H \sum D}{N}$$

$$I = \sum H^2 - \frac{(\sum H)^2}{N}$$

$$K = 3$$

7.3.1.2 Conditioned Parabola

$$A = \sum (H - 1.3)D^2$$

$$D = \sum (H - 1.3)D$$

$$I = \sum (H - 1.3)^2$$

$$K = 2$$

The parabola and conditioned parabola standard error estimate for height are calculated as follows:

$$SEE_h = \sqrt{\frac{I - bD - cA}{N - K}}$$

7.3.1.3 Hyperbola

$$A = \sum HD - \frac{\sum H \sum D}{N}$$

$$D = \sum \frac{H}{D} - \frac{\sum H \sum 1/D}{N}$$

$$I = \sum H^2 - \frac{(\sum H)^2}{N}$$

$$K = 3$$

7.3.1.4 Conditioned Hyperbola

$$A = \sum (H - 1.3)D$$

$$D = \sum \frac{-D(H - 1.3)}{(D + 1)}$$

$$I = \sum (H - 1.3)^2$$

$$K = 2$$

The hyperbola and conditioned hyperbola standard error estimate for height is calculated as follows:

$$SEE_h = \sqrt{\frac{I - bD - cA}{N - K}}$$

7.3.1.5 Weibull and Conditioned Weibull

7.3.1.5.1 Weibull

H = height of tree

H' = calculated height from regression equation

7.3.1.5.2 Conditioned Weibull

H = height of tree

H' = calculated height

$$SEE_h = \sqrt{\frac{\sum(H - H')^2}{N - 3}}$$

7.3.2 Bias - Height

Use the following calculation for all 6 height-diameter equations. The height bias should be zero for the Parabola and Hyperbola.

$$BIAS_h = \frac{\sum(H - H')}{N}$$

Where

H = Height of sample tree in metres, to the nearest tenth

H' = Calculated height in metres, to the nearest tenth

N = Number of sample trees used

7.3.3 Bias - Volume

Use the following calculation for all 6 height-diameter equations.

$$Bias_v = \frac{\sum(v_1 - v_2)}{N}$$

7.3.4 Standard Error Estimate - Volume*

Use the following calculation for all 6 height-diameter equations.

* The equation with the lowest SEE_v will be labeled as the suggested curve.

$$SEE_v = \sqrt{\frac{\sum (v_1 - v_2)^2}{N - K}}$$

Where

$$v_1 = 10^a D_i^b H_i^c$$

$$v_2 = 10^a D_i^b (H_i')^c$$

$$D_i = \text{DBH of sample tree } i$$

$$H_i = \text{Height of sample tree } i$$

$$H_i' = \text{Calculated height of sample tree } i$$

$$a = \text{Volume constant A, from Appendix 15}$$

$$b = \text{Volume constant B, from Appendix 15}$$

$$c = \text{Volume constant C, from Appendix 15}$$

$$N = \text{Number of sample trees used}$$

$$K = 2, \text{ for the conditioned parabola and conditioned hyperbola or}$$

$$K = 3, \text{ for all other functions}$$

7.4.2 Report 004a - Sample Listing

Height curve edits - required for stump cruises.

| FOREST REVENUE BRANCH - 2001.0 | | CALCULATED HEIGHTS OF THE FOLLOWING TREES MAYBE IN ERROR | | | | | | | | | | (004a) | PAGE | 1 |
|--------------------------------|------------------|--|---------|--------|----|-----|--------|--------|-----|----------|------|----------|------------|----------|
| NOT FOR APPRAISAL PURPOSES | | LICENCES: DSC000 - 000 | | | | | | | | | | COMPUTED | 2001-03-19 | 17:45:17 |
| TIMBER TYPE | BLOCK/STRIP/PLOT | TREE NO. | SPECIES | D.B.H. | TC | CAL | HEIGHT | REASON | DBH | GT | TREE | DBH | | |
| 1-1 | 001/01/01 | 63 | F-H | 102.2 | -1 | | 35.2 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 66 | F-H | 103.7 | -1 | | 35.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 41 | F-F | 104.0 | -1 | | 35.6 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 2 | F-F | 106.6 | -1 | | 36.3 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 68 | F-H | 107.7 | -1 | | 36.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 59 | F-F | 111.4 | -1 | | 37.4 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 70 | F-F | 116.3 | -1 | | 38.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 89 | F-F | 124.6 | -1 | | 40.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 12 | F-F | 128.5 | -1 | | 41.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 76 | F-F | 137.6 | -1 | | 43.7 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 97 | F-F | 145.3 | -1 | | 45.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 7 | F-F | 158.1 | -5 | | 48.6 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 11 | F-F | 159.1 | -1 | | 48.8 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 95 | F-F | 166.2 | -1 | | 50.5 | DBH | GT | SAMPLE | TREE | DBH | - | 101.1 |
| 1-1 | 001/01/01 | 13 | F-F | 217.5 | -5 | | 51.5 | DBH | GT | SAMPLE | TREE | DBH | - | 170.5 |
| 1-1 | 001/01/01 | | F-F | 276.8 | -5 | | 51.5 | DBH | GT | EQUATION | MAX | | - | 170.5 |

Figure 7.2 Height Curve Edits - Sample Listing (Partial).

7.4.3 Report 003 - Sample Listing

This report graphs the calculated height vs DBH.

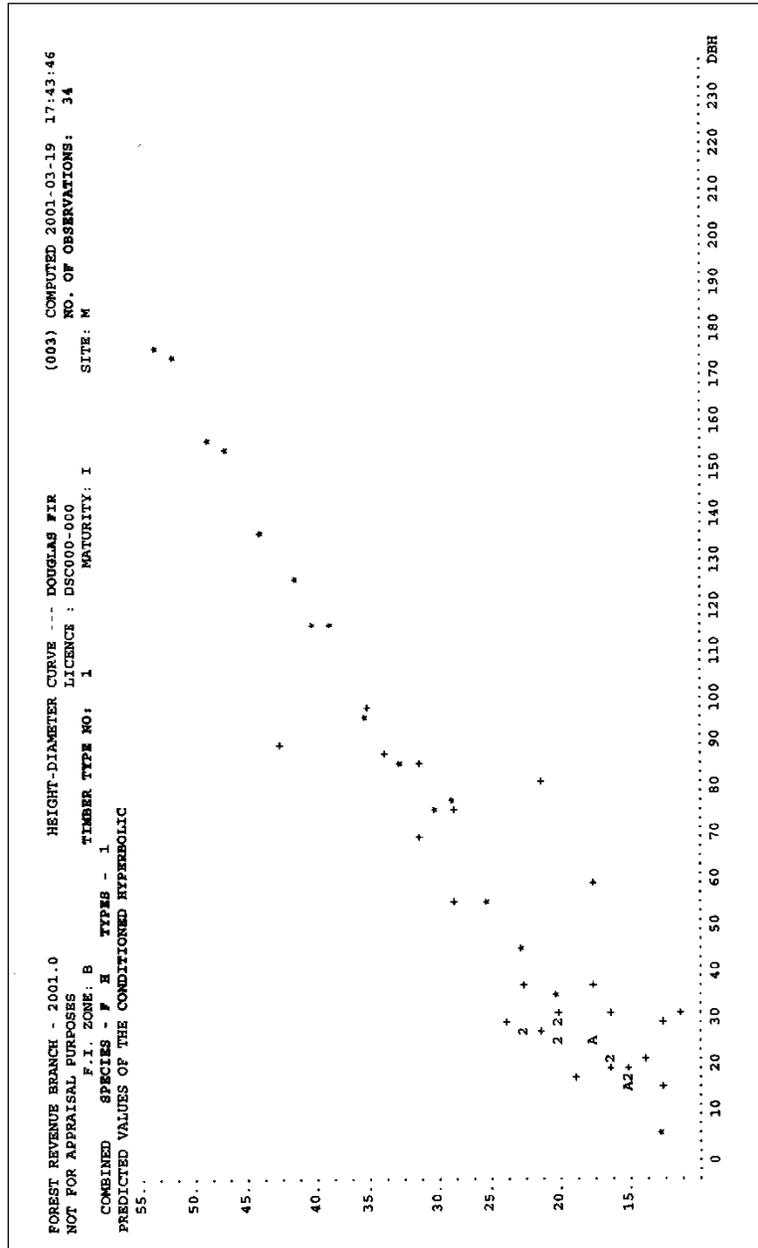


Figure 7.3 Regression Coefficients and Calculated Heights – Sample Listing (Partial).

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