CURRICULUM

Calculus 12

ESTIMATED INSTRUCTIONAL TIME

Calculus 12 has been developed assuming that teachers have 100 instructional hours available to them. The following chart shows the estimated instructional time for each curriculum suborganizer, expressed as a percentage of total time available to teach the course.

Calculus 12

Organizer (Suborganizer)	% of Time
Problem Solving	**Integrated Throughout**
Overview and History of Calculus	**Integrated Throughout**
Functions, Graphs, and Limits (Functions and their Graphs)(Limits)	**10 - 15**
The Derivative (Concept and Interpretations)	**10 - 15**
The Derivative (Computing Derivatives)	**15 - 20**
Applications of Derivatives (Derivatives and the Graph of the Function)	**15 - 20**
Applications of Derivatives (Applied Problems)	**20 - 25**
Antidifferentiation (Recovering Functions from their Derivatives)	**5 - 10**
Antidifferentiation (Applications of Antidifferentiation)	**10 - 15**

When delivering the prescribed curriculum, teachers may freely adjust the instructional time to meet their students' diverse needs. These estimated instructional times have been recommended by the IRP writers to assist their colleagues; they are suggestions only.

PRESCRIBED LEARNING OUTCOMES

It is expected that students will use a variety of methods to solve real-life, practical, technical, and theoretical problems.

It is expected that students will:

- solve problems that involve a specific content area (e.g., geometry, algebra, trigonometry statistics, probability)
- solve problems that involve more than one content area
- solve problems that involve mathematics within other disciplines
- analyse problems and identify the significant elements
- develop specific skills in selecting and using an appropriate problem-solving strategy or combination of strategies chosen from, but not restricted to, the following:
  - guess and check
  - look for a pattern
  - make a systematic list
  - make and use a drawing or model
  - eliminate possibilities
  - work backward
  - simplify the original problem
  - develop alternative original approaches
  - analyse keywords
- demonstrate the ability to work individually and co-operatively to solve problems
- determine that their solutions are correct and reasonable
- clearly communicate a solution to a problem and justify the process used to solve it
- use appropriate technology to assist in problem solving

SUGGESTED INSTRUCTIONAL STRATEGIES

Problem solving is a key aspect of any mathematics course. Working on problems can give students a sense of the excitement involved in creative and logical thinking. It can also help students develop transferable real-life skills and attitudes. Problems may come from various fields, including algebra, geometry, and statistics. Multi-strand and interdisciplinary problems should be included throughout Calculus 12.

- Introduce new types of problems directly to students (without demonstration) and play the role of facilitator as they attempt to solve such problems.
- Recognize when students use a variety of approaches (e.g., algebraic and geometric solutions); avoid becoming prescriptive about approaches to problem solving.
- Reiterate that problems might not be solved in one sitting and that “playing around” with the problem-revisiting it and trying again-is sometimes needed.
- Frequently engage small groups of students (three to five) in co-operative problem solving when introducing new types of problems.
- Have students or groups discuss their thought processes as they attempt a problem. Point out the strategies inherent in their thinking (e.g., guess and check, look for a pattern, make and use a drawing or model).
- Ask leading questions such as:
  - What are you being asked to find out?
  - What do you already know?
  - Do you need additional information?
  - Have you ever seen similar problems?
  - What else can you try?
- Once students have arrived at solutions to particular problems, encourage them to generalize or extend the problem situation.

Suggested Assessment Strategies

Students analyse problems and solve them using a variety of approaches. Assessment of problem-solving skills is made over time, based on observations of many situations.

Observe
- Have students present solutions to the class individually, in pairs, or in small groups. Note the extent to which they clarify their problems and how succinctly they describe the processes used.

Question
- To check the approaches students use when solving problems, ask questions that prompt them to:
  - paraphrase or describe the problem in their
  - own words
  - explain the processes used to derive an answer
  - describe alternative methods to solve a problem
  - relate the strategies used in new situations
  - link mathematics to other subjects and to the world of work

Collect
- On selected problems, have students annotate their work to describe the processes they used. Alternatively, have them provide brief descriptions of what worked and what did not work as they solved particular problems.

Self-Assessment
- Ask students to keep journals to describe the processes they used in dealing with problems. Have them include descriptions of strategies that worked and those that did not.
- Develop with students a set of criteria to self-assess problem-solving skills. The reference set Evaluating Problem Solving Across Curriculum may be helpful in identifying such criteria.

Recommended Learning Resources

Please see the introduction to Appendix B for a list of suggested utility software that supports this course.

The Western Canadian Protocol Learning Resource Evaluation Process has also identified numerous *teacher resources* and *professional references*. These are generally cross-grade planning resources that include ideas for a variety of activities and exercises.

These resources, while not part of the Grade Collections, have Provincially Recommended status.

Appendix B includes an annotated bibliography of these resources for ordering convenience.

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand that calculus was developed to help model dynamic situations.

It is expected that students will:

- distinguish between static situations and dynamic situations
- identify the two classical problems that were solved by the discovery of calculus:
  - the tangent problem
  - the area problem
- describe the two main branches of calculus:
  - differential calculus
  - integral calculus
- understand the limit process and that calculus centers around this concept

SUGGESTED INSTRUCTIONAL STRATEGIES

Students will quickly come to realize that calculus is very different from the mathematics they have previously studied. Of greatest importance is an understanding that calculus is concerned with change and motion. It is a mathematics of change that enables scientists, engineers, economists, and many others to model real-life, dynamic situations.

- The concepts of average and instantaneous velocity are a good place to start. These concepts could be related to the motion of a car. Ask students if the displacement of the car is given by $f(t)$, what is the average velocity of the car between times $t = t_1$ and $t = t_2$, and the instantaneous velocity at $t = t_1$? Point out that calculus is not needed to determine the average velocity $\frac{f(t_2) - f(t_1)}{t_2 - t_1}$, but is needed to determine the velocity at t_1 (the speedometer reading).
- As a way of solving the tangent problem (i.e., the need for two points and the fact that only one point is available), create a secant line that can be moved toward the tangent line. Ask students if they can move the secant line towards the tangent line using algebraic techniques.
- Ask students to determine the area under a curve by estimating the answer using rectangles. Introduce the concept of greater accuracy when the rectangles are of smaller and smaller widths (the idea of limit).
- Present the following statement for discussion:
  $$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$
  Use geometry to illustrate the relationship between the left-hand and right-hand sides.

SUGGESTED ASSESSMENT STRATEGIES

To demonstrate their achievement of the outcomes for this organizer, students need opportunities to engage in open-ended activities that allow for a range of responses and representations.

Although formative assessment of students' achievement of the outcomes related to this organizer should be ongoing, summative assessment can only be carried out effectively toward the end of the course, when students have sufficient understanding of the details of calculus to appreciate the "big picture."

Self-Assessment

- Work with the students to develop criteria and rating systems they can use to assess their own descriptions of the two main branches of calculus. They can represent their progress with symbols to indicate when they have addressed a particular criterion in their work. Appropriate criteria might include the extent to which they:
  - make connections to other branches of mathematics
  - demonstrate understanding of the classical problems
  - consider the different attempts to solve the classical problem(s)
  - explain the significance of limits with reference to specific examples
  - accurately and comprehensively describe the relationship between integral and differential calculus (methodology and purpose)

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  pp. 83-85, 95, 190, 262, 272-273,375
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 57 - 61, 63
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 2 (Section 2.2)
  pp. 3-4, 6-9, 85, 351

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 2, 112, 114, 375
- Calculus of a Single Variable, Sixth Edition
  Ch. 1 (Section 1.1)
  Ch. 4 (Section 4.2, 4.3
  pp. 41-45, 91, 265

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand the historical background and problems that lead to the development of calculus.

It is expected that students will:

- describe the contributions made by various mathematicians and philosophers to the development of calculus, including:
  - Archimedes
  - Fermat
  - Descartes
  - Barrow
  - Newton
  - Leibniz
  - Jakob and Johann Bernoulli
  - Euler
  - L'Hospital

SUGGESTED INSTRUCTIONAL STRATEGIES

Students gain a better understanding and appreciation for this field of mathematics by studying the lives of principal mathematicians credited for the invention of calculus, including the period in which they lived and the significant mathematical problems they were attempting to solve.

- Conduct a brief initial overview of the historical development of calculus, then deal with specific historical developments when addressing related topics (e.g., the contributions of Fermat and Descares to solving the tangent line problem can be covered when dealing with functions, graphs, and limits). For additional ideas, see the suggested instructional strategies for other organizers.
- Encourage students to access the Internet for information on the history of mathematics.
- Ask students to investigate various mathematicians and the associated periods of calculus development and to present their findings to the class.
- Point out the connection between integral and differential calculus when students are working on the "area under the curve" problem.
- Conduct a class discussion about the contributions of Leibniz and Newton when the derivative is being introduced (e.g., the notation $\frac{dy}{dx}$ for the derivative and $\int y\,dx$ for the integral are due to Leibniz; the $f'(x)$ notation is due to Lagrange).
- Have students research the historical context of the applications of antidifferentiation. For example:
  - The area above the x axis, under $y = x^2$, from $x = 0$ to $x = b$ is $b^3/3$. Archimedes was able to show this without calculus, using a difficult argument.
  - The area problem for $y = \frac{1}{x}$ was still unsolved in the early 17th century. Using antiderivatives, the area under the curve $y = \frac{1}{t}$ from $t = 1$ to $t = x$ is $\ln x$.
  - The area under $y = \sin x$, from $x = 0$ to $x = \frac{\pi}{2}$, was calculated by Roberval, without calculus, using a difficult argument. It can be solved easily using antiderivatives.

Suggested Assessment Strategies

When students are aware of the outcomes they are responsible for and the criteria by which their work will be assessed, they can represent their comprehension of the historical development of calculus more creatively and effectively.

Observe

- Have students construct and present a calculus timeline, showing important discoveries and prominent mathematicians. Check the extent to which students' work is:
  - accurate
  - complete
  - clear

Collect

- Ask students to write a short article about one mathematician's contributions to calculus. The article should describe the mathematical context within which the person lived and worked and the contributions as seen in calculus today.

Research

- Use a research assignment to assess students' abilities to synthesize information from more than one source. Have each student select a topic of personal interest, develop a list of three to five key questions, and locate relevant information from at least three different sources. Ask students to summarize what they learn by responding to each of the questions in note form, including diagrams if needed. Look for evidence that they are able to:
  - combine the information, avoiding duplications or contradictions
  - make decisions about which points are most important

Peer Assessment

- To check on students' knowledge of historical figures, form small groups and ask each group to prepare a series of three to five questions about the contribution of a particular mathematician. Have groups exchange questions, then discuss, summarize, and present their answers. For each presentation, the group that designed the questions offers feedback on the extent to which the answers are thorough, logical, relevant, and supported by specific explanation of the mathematics involved.

Recommended Learning Resources

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Integrated Throughout
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  Integrated Throughout
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Integrated Throughout

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 10-11, 15, 19, 99, 225, 237, 278, 352, 378, 638, A3
- Calculus of a Single Variable, Sixth Edition
  Integrated Throughout

Prescribed Learning Outcomes

It is expected that students will represent and analyse rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions, using technology as appropriate.

It is expected that students will:

- model and apply inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions to solve problems
- draw (using technology), sketch and analyse the graphs of rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions, for:
  - domain and range
  - intercepts
- recognize the relationship between a base a exponential function ($a > 0$) and the equivalent base e exponential function (convert $y = a^x$ to $y = e^{x(1na)}$)
- determine, using the appropriate method (analytic or graphing utility) the points where $f(x) = 0$

Suggested Instructional Strategies

With the aid of technology, graphs of functions are often easy to produce. The emphasis is on the interplay between the geometric and analytic information and on the use of calculus both to predict and to explain the observed local and global behavior of a function.

- Prior to undertaking work on functions and their graphs, conduct review activities to ensure that students can:
  - perform algebraic operations on functions and compute composite functions
  - use function and inverse function notation appropriately
  - determine the inverse of a function and whether it exists
  - describe the relationship between the domain and range of a function and its inverse
- Give students an exponential function and its inverse logarithmic function. Have them work in groups, using graphing calculators or computer software, to identify the relationship between the two. (e.g., compare and contrast the graphs of $\log_2 x$ and 2^x).
- Emphasize the domain and range restrictions as students work with exponential and natural logarithmic functions.
- Have students use technology (e.g., graphing calculators) to compare the graphs of functions such as:

  $y = e^x$ and $y = \ln x$

  $y = e^{\ln x}$ and $y = \ln e^x$

  $y = 3^x$ and $y = e^{x \ln 3}$
- Similarly apply this procedure to trigonometric and inverse trigonometric graphs including: sine and arcsine, cosine and arccosine, and tangent and arctangent. Emphasize that arcsine $x = \sin^{-1} x$ and does not mean the reciprocal of sin x.

SUGGESTED ASSESSMENT STRATEGIES

The exponential and natural logarithmic functions enable students to solve more complex problems in areas such as science, engineering, and finance. Students should be able to demonstrate their knowledge of the relationship between natural logarithms and exponential functions, both in theory and in application, in a variety of problem situations.

Observe

- Ask students to outline how they would teach a classmate to explain:
  - the inverse relationship between base *e* exponential and natural logarithmic functions
  - the restrictions associated with base *e* exponential and natural logarithmic, and trigonometric and inverse trigonometric functions

  Note the extent to which the outlines:
  - include general steps to follow
  - use mathematical terms correctly
  - provide clear examples
  - describe common errors and how they can be avoided

Collect

- Assign a series of problems that require students to apply their knowledge of the relationship between logarithms and exponential functions. Check their work for evidence that they:
  - clearly understood the requirements of the problem
  - used efficient strategies and procedures to solve the problem
  - recognized when a strategy or procedure was not appropriate
  - verified that their solutions were accurate and reasonable

Self-Assessment

- Discuss with students the criteria for assessing graphing skills. Show them how to develop a rating scale. Have them use the scale to assess their own graphing skills.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  pp. 12, 16, 23, 36-37, 45-51, 165
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  Ch. 1 (Sections 1.4, 1.5, 1.6)
  Ch. 4 (Section 4.6)
  pp. 5, 20, 26-27, 41, 51-52, 63, 158, 182
- Single Variable Calculus Early Transcendentals, Fourth Edition
  pp. 43-46, 61-63, 69-72, A31-A34

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 49, 57, 240, 242-243, 246, 254, 261, 308
- Calculus of a Single Variable, Sixth Edition
  Ch. 5 (Section 5.1, 5.4, 5.5, 5.8)
  pp. 20, 26, 27, 75, 134, 380

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand the concept of limit of a function and the notation used, and be able to evaluate the limit of a function.

It is expected that students will:

- demonstrate an understanding of the concept of limit and notation used in expressing the limit of a function $f(x)$ as x approaches a: $\lim_{x\to a} f(x)$
- evaluate the limit of a function
  - analytically
  - graphically
  - numerically
- distinguish between the limit of a function as x approaches a and the value of the function at $x = a$
- demonstrate an understanding of the concept of one-sided limits and evaluate one-sided limits
- determine limits that result in infinity (infinite limits)
- evaluate limits of functions as x approaches infinity (limits at infinity)
- determine vertical and horizontal asymptotes of a function, using limits
- determine whether a function is continuous at $x = a$

SUGGESTED INSTRUCTIONAL STRATEGIES

Students require a firm understanding of limits in order to fully appreciate the development of calculus.

- Although the concept of limit should be introduced at the beginning of the course, particular limits (e.g., $\lim_{x\to 0}\frac{\sin x}{x}$) need only be introduced as needed to deal with new derivatives.
- Using diagrams, introduce students in a general way to the two classic problems in calculus: the **tangent line problem** and the **area problem** and explain how the concept of limit is used in the analysis.
- Give students a simple function such as $f(x) = x^2$ and have them:
  - determine the slope of the tangent line to $f(x) = x^2$ at the point (2,4) by evaluating the slope of the secant lines through (2,4) and (2.5, f(2.5)), (2.1, f(2.1)), (2.05, f(2.05)), (2.01, f(2.01)), then draw conclusions about the value of the slope of the tangent line
  - determine the area under $y = x^2$ above the x axis from $x = 0$ to $x = 2$ by letting the number of rectangles be 4, 8, and 16, then inferring the exact area under the curve.
- Give students limit problems that call for analytic, graphical, and/or numerical evaluation, as in the following examples:
  - $\lim_{x\to 2}\frac{x^2-4}{x-2}$ (evaluate analytically)
  - $\lim_{x\to 0}\frac{\sin x}{x}$ (evaluate numerically, geometrically, and using technology)
  - $\lim_{x\to 0}\frac{3x^2+5}{4-x^2}$ (evaluate analytically and numerically)
  - $\lim_{x\to 0}\frac{1}{x^2}=\infty$ (draw conclusions numerically)
- When introducing students to one-sided limits that result in infinity, have them draw conclusions about the vertical asymptote to $y = f(x)$, as in the following example: from $\lim_{x\to 3^+}\frac{1}{x-3}$ and $\lim_{x\to 3^-}\frac{1}{x-3}$, conclude that $x = 3$ is the vertical asymptote for $f(x)=\frac{1}{x-3}$.
- Have students explore the limits to infinity of a function and draw conclusions about the horizontal asymptote, as in the following example: because $\lim_{x\to\infty}\frac{e^x}{e^x+1}=1$ and because $\lim_{x\to -\infty}\frac{e^x}{e^x+1}=0$, y=1 and y=0 are horizontal asymptotes for $f(x)=\frac{e^x}{e^x+1}$.

SUGGESTED ASSESSMENT STRATEGIES

Limits form the basis of how calculus can be used to solve previously unsolvable problems. Students should be able to demonstrate their knowledge of both the theory and application of limits in a variety of problem situations.

Observe

- While students are working on problems involving limits, look for evidence that they can:
  - distinguish between the limit of a function and the **value** of a function
  - Identify and evaluate one-sided limits
  - recognize and evaluate infinite limits and limits at infinity
  - determine any vertical or horizontal asymptote of a function
  - determine whether a function is continuous over a specified range or point
- To check students' abilities to reason mathematically, have them describe orally the characteristics of limits in relation to one-sided limits, infinite limits, limits at infinity, asymptotes, and continuity of a function. To what extent do they give explanations that are mathematically correct, logical, and clearly presented.

Collect

- Assign a series of problems that require students to apply their knowledge of limits. Check their work for evidence that they:
  - clearly understood the requirements of the problem
  - used the appropriate method to evaluate the limit in the problem
  - verified that their solutions were accurate and reasonable

Self-Assess/Peer Assess

- Have students explain concepts such as "limit" and "continuity" to each other in their own words.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic pp. 55-61, 65-77
- Calculus of a Single Variable Early Transcendental Functions, Second Edition pp. 63, 72, 75, 84, 96-97, 113, 227-228
- Single Variable Calculus Early Transcendentals, Fourth Edition Ch. 2 (Sections 2.2, 2.5, 2.6)

Multimedia

- Calculus: A New Horizon, Sixth Edition pp. 115, 118, 120, 122-124, 131, 148, 161
- Calculus of a Single Variable, Sixth Edition Ch. 1 (Section 1.2, 1.4, 1.5) Ch. 3 (Section 3.5)

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

It is expected that students will:

- describe geometrically a secant line and a tangent line for the graph of a function at $x = a$
- define and evaluate the derivative at $x = a$ as: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
- define and calculate the derivative of a function using:
  $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or
  $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ or
  $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- use alternate notation interchangeably to express derivatives (i.e., $f'(x), \frac{dy}{dx}, y'$, etc.)
- compute derivatives using the definition of derivative
- distinguish between continuity and differentiability of a function at a point
- determine when a function is non-differentiable, and explain why
- determine the slope of a tangent line to a curve at a given point
- determine the equation of the tangent line to a curve at a given point
- for a displacement function $s = s(t)$, calculate the average velocity over a given time interval and the instantaneous velocity at a given time
- distinguish between average and instantaneous rate of change

SUGGESTED INSTRUCTIONAL STRATEGIES

The concept of the derivative can be used to calculate instantaneous rate of change. Students can best understand this concept if it is presented to them geometrically, analytically, and numerically. This allows them to make connections between the various forms of presentation.

- Present an introduction to the "tangent line" problem by discussing the contribution made by Fermat, Descartes, Newton, and Leibniz.
- Use a graphing utility to graph $f(x) = 2x^3 - 4x^2 + 3x - 5$.
  On the same screen, add the graphs of $y = x - 5$, $y = 2x - 5$, $y = 3x - 5$. Have students identify which of these lines appears to be tangent to the graph at the point (0, -5). Have them explain their answers.
- Sketch a simple parabola $y = x^2$ on the board. Attach a string at a point and show how it can be moved from a given secant line to a tangent line at a different point on the graph. Show how the slope of the secant line approaches that of the tangent line as $\Delta x \to 0$.
- Have students calculate the slope of a linear function at a point $x = a$ (e.g., $f(x) = 3x + 2$ at $x = 1$), using the formula $\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$. Have students use the same formula to find the slope of the tangent lines to the graph of a non-linear function such as $f(x) = x^2 - 4$ at $a = 1$, and then find the equation of the tangent lines.
- Ask students to use technology to:
  - reinforce the result found when using the definition to calculate the derivative of a function
  - verify that the tangent line as calculated, appears to be the tangent to the curve. As an extension, have students zoom in at the point of tangency and describe the relationship between the tangent line and the curve
- Introduce the differentiability of a function by generating problems that require students to create, using technology, graphs with sharp turns and vertical tangent lines.
- Have students work in groups to prepare a presentation on tangent lines and the use of mathematics by the ancient Greeks (circle, ellipse, parabola). Ask them to include an explanation of how the initial work of the Greeks was surpassed by the work of Fermat and Descartes and how previously difficult arguments were made into essentially routine calculations.

SUGGESTED ASSESSMENT STRATEGIES

The concept of the derivative facilitates the solving of complex problems in fields such as science, engineering, and finance. Students should be able to demonstrate knowledge of the derivative in relation to problems involving instantaneous rate of change.

Observe

- As students work, circulate through the classroom and note:
  - the extent to which they relate their algebraic work to their graphing work
  - whether they can recognize the declining nature of the x values
- Have students develop a table of values to calculate a "slope predictor" in the parabola $y = x^2$ at a point. In small groups they can justify the concept of a limit using technology. Higher magnifications will eliminate the difference in the slope values of secant and tangent lines at a point. Ask one student from each group to explain this phenomenon, using the group's 'unique' sample slope calculation. Note how succinctly they describe the processes used.
- When students work with technology (e.g., graphing calculator), check the extent to which they:
  - select appropriate viewing windows
  - enter the functions correctly
  - interpret the results correctly

Question

- Have students explain in their own words the concepts of average and instantaneous rates of change.

Research

- When students report on the contributions of various mathematicians to the tangent line problem, check the extent to which they:
  - clearly address key concepts (e.g., explain how Fermat found the length of the subtangent; how Descartes found the slope of the normal)
  - include careful graphic illustrations

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  pp. 83-85, 95-97, 105-107, 122-124, 152
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 60, 110, 112-117, 123, 125-129, 137, 142, 174, 662, 675
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 2 (Sections 2.7, 2.8, 2.9)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  Ch. 3
  pp. 170, 172, 175, 178-180, 186, 187, 353
- Calculus of a Single Variable, Sixth Edition
  Ch. 2 (Section 2.1)
  pp. 92-97, 108-109, 166

PRESCRIBED LEARNING OUTCOMES

It is expected that students will determine derivatives of functions using a variety of techniques.

It is expected that students will:

- compute and recall the derivatives of elementary functions including:
  - $\frac{d}{dx}(x^r) = rx^{r-1}$, r is real
  - $\frac{d}{dx}(e^x) = e^x$
  - $\frac{d}{dx}(\ln x) = \frac{1}{x}$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- use the following derivative formulas to compute derivatives for the corresponding types of functions:
  - constant times a function $\frac{d}{dx}cu = c\frac{du}{dx}$:
  - $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ (sum rule)
  - $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (product rule)
  - $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (quotient rule)
  - $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$ (power rule)
- use the Chain Rule to compute the derivative of a composite function: $\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$ or $\frac{d}{dx}(F(g(x))) = g'(x)F'(g(x))$
- compute the derivative of an implicit function
- use the technique of logarithmic differentiation
- compute higher order derivatives

SUGGESTED INSTRUCTIONAL STRATEGIES

Knowledge of a variety of techniques for computing the derivatives of different types of functions allows students to solve an ever-increasing range of problems. To become efficient problem solvers students need to understand when and how to use derivative formulas, but will also find it useful to memorize some basic derivatives.

- Point out to students that the derivatives of a few functions (e.g., x^2, $\frac{1}{x}$, $\sqrt{x}$) will already have been obtained using the definition of a derivative.
- To help students connect the limit concept to the derivative of sin x, demonstrate the details of $\frac{d}{dx}(\sin x) = \cos x$, using the known fact that $\sin(x + h) = \cos x \sin h + \sin x \cos h$ and that $\lim_{h\to 0}\frac{\sin h}{h} = 1$. Alternatively, you can provide these known facts to students and have them work in groups to develop the solution.
- Have students work in groups to differentiate $f(x) = \frac{(1+x)^5(1-2x)^7}{x^3}$, using two different methods:
  1) the chain, product, and quotient rules
  2) logarithmic differentiation

  Ask students to compare the two methods.
- Demonstrate that the derivatives of functions such as ln x, $\sin^{-1} x$, and $\cos^{-1} x$ can be found using implicit differentiation. For example, given $y = \sin^{-1} x$, $\frac{dy}{dx}$ can be determined implicitly, as follows:
  - Since $y = \sin^{-1} x$, then $\sin y = x$
  - Thus $\cos y\frac{dy}{dx} = x$, which means $\frac{dy}{dx} = \frac{1}{\cos y}$
  - Since $\cos^2 y = 1 - \sin^2 y$, then $\cos y = \sqrt{1-\sin^2 y}$
  - Substituting $\sin y = x$, we get $\cos y = \sqrt{1-x^2}$.
  - Now substitute this into $\frac{dy}{dx} = \frac{1}{\cos y}$ to get $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
- Although full proof of the chain rule is uninformative, a graphing calculator can be used to show quite persuasively that for example $\frac{d}{dx}(\sin 3x)$ ought to be 3 cos 3x at any particular place x [given that $\frac{d}{dx}\sin x = \cos x$].

SUGGESTED ASSESSMENT STRATEGIES

As students develop confidence and proficiency computing the derivative of elementary functions, they are better prepared to solve more complex problems that involve derivative formulas. Assessment should focus on students' ability to recall derivatives of elementary functions and apply this knowledge appropriately when computing the derivative of a more complex function (using formulas, implicit differentiation, or logarithmic differentiation).

Observe

- As students are working on problems, circulate and provide feedback on their notation use. Have students verify their work using a graphing calculator.

Collect

- For a question where $f(x) = (x^2 + x)$ 3 ask students to debate the effectiveness of finding the derivative using expansion, the chain rule, the product rule, and logarithmic differentiation. To what extent do their arguments illustrate their ability to expand on current mathematical idea?
- Have students present to the class research on some elementary functions and their derivatives.
- Discuss with students the merits of the various methods of taking derivatives. Have them summarize in writing their understanding of the merits of each. Work with them to develop a set of criteria to assess the summaries.

Self-Assessment

- To assess students' command of the chain rule, have them list common difficulties they encounter in taking derivatives. Let them work in pairs to develop checklists of reminders to use when they're checking their work.
- Ask students to summarize derivatives of elementary functions, making notes on notation errors. Allow students to complete assignments using their summaries.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 3 (Sections 3.3, 3.5, 3.6, 3.7, 3.8, 3.9)
  pp. 119, 151-154, 169
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 121, 125-126, 133-134, 137, 139, 143, 149, 158, 163, 168, 170
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 3 (Sections 3.1, 3.2, 3.4, 3.5, 3.6, 3.7 3.8)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 189, 191-196, 200, 204, 246, 257, 258, 261
- Calculus of a Single Variable, Sixth Edition
  Ch. 2 (Section 2.5)
  pp. 103, 105-107, 114-120,125-126, 315-316, 340, 380

PRESCRIBED LEARNING OUTCOMES

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

It is expected that students will:

- given the graph of $y = f(x)$:
  - graph $y = f'(x)$ and $y = f''(x)$
  - relate the sign of the derivative on an interval to whether the function is increasing or decreasing over that interval.
  - relate the sign of the second derivative to the concavity of a function
- determine the critical numbers and inflection points of a function
- determine the maximum and minimum values of a function and use the first and/or second derivative test(s) to justify their solutions
- use Newton's iterative formula (with technology) to find the solution of given equations, $f(x) = 0$
- use the tangent line approximation to estimate values of a function near a point and analyse the approximation using the second derivative

SUGGESTED INSTRUCTIONAL STRATEGIES

The first and second derivatives of a function provide a great deal of information concerning the graph of the function. This information is critical to solving problems involving calculus and helps students understand what the graph of a function represents.

- Ask students to compare the questions, "Where is $-x^3 + 14x^2 + 20x$ increasing most rapidly? Where is $-x^3 + 14x^2 + 20x$ increasing?" Have students develop similar questions.
- Have students work in teams of two to determine why the calculator is not helpful in questions such as: "let $f(x) = ax^3 - 5x^2$. Describe in terms of the parameter a, where $f(x)$ reaches a maximum or minimum."
- Have students use technology to explore features of functions such as the following:
  - when a function has maximum/minimum points or neither
  - where the inflection points occur
  - where curves are concave up or down
  - vertical or horizontal tangent lines
  - the relationship among graphs of the 1st, 2nd derivatives of a function and the function
  - the impact that endpoints have on the maximum/minimum
- Use a summary chart to demonstrate increasing and decreasing aspects of a function. Have students describe how this relates to the 1st derivative.
- Have groups of students learn particular concepts, (e.g., Newton's method) and teach these to their peers. Have them examine situations where it works very efficiently, situations where it works very slowly (e.g., $x^5 = 0$), and situations where it does not work (e.g., $x^{\frac{1}{3}} = 0$). Challenge them to develop hypotheses as to why it does or does not work.
- Have students research and report on the history of Newton's method for determining square roots (e.g., mentioning Heron of Alexandria, and the work of mathematicians in Babylon, India).
- Discuss with students the relative merits of using or not using a graphing calculator "to do" calculus. Present them with questions such as $g(x) = 8x^3 - 5x^2 + x - 3$ contrasted with $f(x) = \frac{1}{3}x^3 - 10{,}000x + 100$. Note that the calculator does not always give the necessary detail.

Suggested Assessment Strategies

Students demonstrate their understanding of first and second derivatives by relating the information they obtain from these to the graph of a function (i.e., critical numbers, inflection points, maximum and minimum values, and concavity).

Observe

- When reviewing students work, note the extent to which they can:
  - accurately determine the 1st and 2nd derivative
  - apply the respective tests
  - recognize and describe the relationship among the graphs of f(x), f′(x), f″(x)

Collect

- Ask the students to sketch a cubic graph and determine what the slopes of the tangent line would be at the local maximum and minimum points. Have them present their summary to the class. Consider the extent to which they can explain why the derivative is 0 at a local maximum/minimum.

Peer assessment

- Have students critique each others' graphs using criteria generated by the class. These criteria could include the extent to which:
  - a graph is appropriate to the function it is supposed to represent
  - the axes are accurately labelled
  - appropriate scales have been chosen for the axes
  - the graphs present smooth curves
  - domain, range, asymptotes, intercepts, and vertices have been correctly determined
  - inflection points, maximum and minimum points, and region of concavity have been correctly identified

Recommended Learning Resources

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 4 (Sections 4.4, 4.6)
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 82, 127-128, 173, 247
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 3 (Section 3.10)
  Ch. 4 (Sections 41, 4.7)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 172, 270, 329
  Calculus of a Single Variable, Sixth Edition
  Ch. 2 (Section 2.6)
  Ch. 3 (Section 3.7)

PRESCRIBED LEARNING OUTCOMES

It is expected that students will solve applied problems from a variety of fields including the Physical and Biological Sciences, Economics, and Business.

It is expected that students will:

- solve problems involving displacement, velocity, acceleration
- solve related rates problems
- solve optimization problems (applied maximum/minimum problems)

SUGGESTED INSTRUCTIONAL STRATEGIES

Calculus was developed to solve problems that had previously been difficult or impossible to solve. Such problems include related rate and optimization problems, which arise in a variety of fields that students may be studying (e.g., the physical and biological sciences, economics, and business).

- Have a student demonstrate a "student trip" in front of the class — constant speed, accelerate, stop, slow down, stop, back up. Have students sketch a graph of displacement against time for the movements.
- Discuss average velocity over a specified interval, instantaneous velocity at specified times. Have students in pairs create a graph and have their partner perform the motion.
- Have students brainstorm examples of the need for calculus in the real world:
  - population growths of bacteria
  - the optimum shape of a container
  - water draining out of a tank
  - the path that requires the least time to travel
  - marginal cost and profit
- Challenge students to create their own "new" problems, which they must then try to solve. These problems could be used to develop tests or unit reviews.
- Have students use technology (e.g., graphing calculators) to investigate problems and confirm their analytical solutions graphically.
- Discuss with students the merits of using versus not using a graphing calculator "to do" calculus.

SUGGESTED ASSESSMENT STRATEGIES

Students develop their knowledge of derivatives by solving problems involving rates of change, maximum, and minimum. When assessing student performance in relation to these problems, it is important to consider students' abilities to make generalizations and predictions about how calculus is used in the real world.

Observe

- While students are working on problems involving derivatives, look for evidence they:
  - clearly understand the requirements of the problem
  - recognized when a strategy was not appropriate
  - can explain the process used to determine their answers
  - used a graphing calculator where appropriate to help visualize their solution
  - verified that their solutions where correct and reasonable
- Provide a number of problems where students are required to use average velocity **or** instantaneous velocity. Observe the extent to which students are able to:
  - determine whether the situation calls for the calculation of average velocity or instantaneous velocity
  - explain the differences between the two
  - provide other examples

Collect

- Assign a series of problems that require students to apply their knowledge of the dynamics of change: how, at certain time the speed of an object is related to its height, and how, at a certain time the speed of an object is related to the change in velocity.
- To discover how well students can recognize and explain the concepts of change and rapid change give them a problem such as:

  A bacterial colony grows at a rate $\frac{dy}{dt} = y(C - y)$.
  How large is the colony when it is growing most rapidly? In their analysis look for evidence that they understand the rate of change of "the rate of change."

Question

- While students are working on simple area problems based on real-life applications, ask them to explain the relationship between the graph's units (on each axis) and the units of area under the curve.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 4 (Sections 4.3, 5.4)
  pp. 97-98, 172, 180, 198, 202
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 182, 184, 195, 197, 209, 219-221, 247, 257
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 2 (Section 2.9)
  Ch. 3 (Section 3.11)
  Ch. 4 (Sections 4.2, 4.3, 4.9)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 211, 290, 299, 363
- Calculus of a Single Variable, Sixth Edition
  Ch. 3 (Sections 3.2, 3.3, 3.8, 3.9)
  pp. 157, 171-175, 182-183

PRESCRIBED LEARNING OUTCOMES

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

It is expected that students will:

- explain the meaning of the phrase "F(x) is an antiderivative (or indefinite integral) of f(x)"
- use antiderivative notation appropriately (i.e., $\int f(x)dx$ for the antiderivative of $f(x)$)
- compute the antiderivatives of linear combinations of functions whose individual antiderivatives are known including:
  - $\int k\,dx = kx + C$
  - $\int x^r dx = \frac{x^{r+1}}{r+1} + C$ if $r \neq -1$
  - $\int \frac{dx}{x} = \ln|x| + C$
  - $\int e^x dx = e^x + C$
  - $\int \sin x\,dx = -\cos x + C$
  - $\int \cos x\,dx = \sin x + C$
  - $\int \sec^2 x\,dx = \tan x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
  - $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- compute $\int f(ax+b)\,dx$ if $\int f(u)\,du$ is known
- create integration formulas from the known differentiation formulas
- solve initial value problems using the concept that if $F'(x) = G'(x)$ on an interval, then $F(x)$ and $G(x)$ differ by a constant on that interval

SUGGESTED INSTRUCTIONAL STRATEGIES

Students need to recognize that antidifferentiation is the reverse of the differentiation process. This understanding will enable them to calculate or verify.

- Use a variety of methods to explain the need for the constant, "C":
  - computational: $\frac{dy}{dx}(C) = 0$
  - geometrical: if we draw curves y = F(x), y = F(x) + C, one curve is obtained from the other by simply lifting, so slopes of tangent lines match
  - kinematic: if v(t) (velocity) is specified, then $\int v(t)dt$ represents all possible position (displacement) functions (we can't reconstruct position from velocity unless we know where we were at some time)
- To ensure students understand the arbitrary nature of assigning letters to variables, use a reasonable variety of letters in doing integration problems. For example, $\int \sin 2u\,du$, $\int e^{at}dt$.
- Integrals such as $\int (1+2x)^9 dx$ or $\int 7e^{x/4} dx$ can be calculated by a "guess and check" process; there is no need at this stage for a substitution rule of integrals. For example, to find $\int 5\sin 6x\,dx$, a student might "guess" that an answer is cos 6x, check by differentiating to get -6 sin 6x, then adjust to determine that the indefinite integral is $-\frac{5}{6}\cos 6x + C$.
- Initial value problems can be treated very informally. For example, suppose that f′(x) = e^{2x} and f(1) = 3. What is f(x)? One antiderivative of e^{2x} is $\frac{e^{2x}}{2}$ (check by differentiating), but this function does not satisfy the initial condition. To fix it, students can simply remember that the general antiderivative is $\frac{e^{2x}}{2} + C$, then choose C appropriately.
- Play a game by challenging students to write down as fast as possible antiderivatives of simple functions, such as $\int$ f(au + b)du where $\int$ f(x)dx is one of the standard integrals obtained by reversing familiar differentiation results.
- Introduce students to integration tables or software applications for symbolic integration.

SUGGESTED ASSESSMENT STRATEGIES

In making connections between antidifferentiation and differentiation, students should relate the physical world to abstract calculus representations. To assess students' learning, it is appropriate to consider their oral and written explanations of the meaning of antidifferentiation as well as evidence from their computation, their drawing, and their problem-solving activities.

Peer Assessment

- Have students work in pairs to find antidifferentiation problems on the Internet (e.g., at university mathematics department web pages), try to solve the problems they find, and verify their solutions.

Observe/Question

- Ensure students' notation is correct at all times (e.g., $\int e^{3x} dx = \frac{e^{3x}}{3} + C$). It is not necessary to try to justify the dx part of the notation, but the C (constant) part should be understood. Use questions to check understanding, such as "Can you think of a function whose derivative is x^2?"
- Divide the students into small groups, and give them a problem whose conventional solution uses a technique that they have not learned, (e.g., general substitution or integration by parts). For example, ($\int xe^{3x}\,dx$, or $\int x^2 \ln x \, dx$. Observe the degree of working control students have of antiderivatives (and derivatives) by seeing how efficiently they can experiment their way to an answer (i.e., generate a variety of plausible ideas).

Collect

- Collect samples of students' worksheets dealing with recovery of functions from their derivatives. Assess the extent to which students are able to compute the antiderivation of functions and determine the value of C, when given initial conditions.

Self/Peer Assessment

- Have students make up tests or quizzes on antidifferentiation techniques and exchange their tests with partners. The partners should check each others' work. Allow the partners time to discuss the work and help each other formulate a plan to address areas of weakness.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 6 (Sections 6.1, 6.2)
  pp. 190, 304 - 307
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 278, 279-282, 319, 325, 342, 347, 374, 490, 495
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 4 (Section 4.10)
  Ch. 5 (Section 5.4, 5.5)
  p. 585

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 382, 384, 388, 392, 581
  Calculus of a Single Variable, Sixth Edition
  pp. 241, 243, 246, 290-291, 366, 385

PRESCRIBED LEARNING OUTCOMES

It is expected that students will use antidifferentiation to solve a variety of problems.

It is expected that students will:

- use antidifferentiation to solve problems about motion of a particle along a line that involve:
  - computing the displacement given initial position and velocity as a function of time
  - computing velocity and/or displacement given suitable initial conditions and acceleration as a function of time
- use antidifferentiation to find the area under the curve $y = f(x)$, above the x-axis, from $x = a$ to $x = b$
- use differentiation to determine whether a given function or family of functions is a solution of a given differential equation
- use correct notation and form when writing the general and particular solution for differential equations
- model and solve exponential growth and decay problems using a differential equation of the form: $\frac{dy}{dt} = ky$
- model and solve problems involving Newton's Law of Cooling using a differential equation of the form: $\frac{dy}{dt} = ay + b$

SUGGESTED INSTRUCTIONAL STRATEGIES

Antidifferentiation enables students to solve problems that are otherwise unsolvable (e.g., problems related to exponential growth and decay). These problems are found in a variety of contexts, from science to business.

- Demonstrate how physics formulae about motion under constant acceleration can be justified using antidifferentiation. If a(t) = -g, then v(t) = -gt + C for some C, and therefore $s(t) = -\frac{1}{2}gt^2 + Ct + D$ for some D. The two constants of integration can be found by using initial conditions. Given g = 9.81 and a rock thrown upward at a given speed from a 100 m tower, it is possible to calculate the maximum height reached and the time it takes for the rock to hit the ground. There are many variations on this problem.
- To explain why antiderivatives can be used to solve area problems, let f(x) be some given function, and let A(u) be the area under y = f(x), above the x axis, from x=a to x=u. By looking at $\frac{A(u+h) - A(u)}{h}$, we can argue reasonably that A′(u) = f(u), at the same time reinforcing the concept of derivative from the definition. So A(u) is an antiderivative of f(u). A(a)=0.
- Many calculators have a numerical integration feature that approximates $\int_a^b f(x)dx$. Have students find the area from a to b under a curve y=f(x) for which they can find $\int$ f(x)dx. Compare this solution with the calculator's approximation.
- Give students a function such as f(x) = ln x and ask them to make a rough sketch of an antiderivative F(x) of f(x) such that F(1) = 0.
- Illustrate the wide applicability of the concepts by using examples that are not from the physical sciences. For example, let C(x) be the cost of producing x tons of a certain fertilizer. Suppose that (for reasonable x), C′(x) is about 30 - 0.02x. The cost of producing 2 tons is $5000. What is the cost of producing 100 tons?
- Have students generate and maintain a list of phenomena other than radioactive decay that are described by the same differential equation. For example, the illumination I(x) that reaches x metres below the surface of the water can be described by $\frac{dI}{dx} = -kI$. Under constant inflation, the buying power V(t) of a dollar t years of now can be described by $\frac{dV}{dt} = -kV$.

SUGGESTED ASSESSMENT STRATEGIES

An understanding of antidifferentiation and its applications is essential to the solution of the "area under the curve" problems. Students can demonstrate their understanding of antidifferentiation ideas and skills through their problem-solving work.

- To check whether the meaning of the phrase "solution of a differential equation" is fully understood, have students work in groups to discuss and solve problems such as the following:
  - show that $y = x^8$ is a solution of the differential equation $x\frac{dy}{dx} = 8y$
  - find a solution of the above differential equation with $y(2) = 16$
- When students are asked to solve problems such as the following, verify that they are able to identify sin *kt* and cos *kt* as solutions of the differential equation and that they can find other solutions as well:
  - A weight hangs from an ideal spring. It is pulled down a few centimetres then released. If y is the displacement of the weight from its rest position, it turns out that $\frac{d^2y}{dt^2} = -k^2y$ for some constant k
- Take a problem (e.g., an exponential decay problem with several parts) and ask for the solution to be written up as a prose report, in complete sentences, with the reasons for each step clearly explained. Criteria for assessment include clarity and grammatical correctness.
- Pose the problem, "In how many different ways can we find the area under $y = x^2$, above the x axis, from $x = 0$ to $x = a$, exactly or approximately?" Students can write a report on this. Students should be able to identify
  - Archimedes' method
  - standard antiderivative method
  - approximation techniques of their own devising
- Ask students to use standard Internet resources to find information about differential equations in various areas of application, and to report on an area of interest to them. Assess the extent to which they are able to express their findings coherently and in their own words.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 6 (Section 6.1, 6.4)
  pp. 262 - 263, 311, 333-334
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 278, 290, 373-374, 377, 381-382, 385, 409
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 4 (Section 4.10)
  Ch. 5 (Sections 5.1, 5.4)
  pp. 586, 603, 611

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 328, 408, 433, 580, 601, 611
- Calculus of a Single Variable, Sixth Edition
  Ch. 4 (Section 4.4)
  Ch. 5 (Section 5.5)
  pp. 242, 362

Prescribed Learning Outcomes

▶ Problem Solving

It is expected that students will use a variety of methods to solve real-life, practical, technical, and theoretical problems.

It is expected that students will:

- solve problems that involve a specific content area (e.g., geometry, algebra, trigonometry statistics, probability)
- solve problems that involve more than one content area
- solve problems that involve mathematics within other disciplines
- analyse problems and identify the significant elements
- develop specific skills in selecting and using an appropriate problem-solving strategy or combination of strategies chosen from, but not restricted to, the following:
  - guess and check
  - look for a pattern
  - make a systematic list
  - make and use a drawing or model
  - eliminate possibilities
  - work backward
  - simplify the original problem
  - develop alternative original approaches
  - analyse keywords
- demonstrate the ability to work individually and co-operatively to solve problems
- determine that their solutions are correct and reasonable
- clearly communicate a solution to a problem and justify the process used to solve it
- use appropriate technology to assist in problem solving

▶ Overview and History of Calculus *(Overview of Calculus)*

It is expected that students will understand that calculus was developed to help model dynamic situations.

It is expected that students will:

- distinguish between static situations and dynamic situations.
- identify the two classical problems that were solved by the discovery of calculus:
  - the tangent problem
  - the area problem
- describe the two main branches of calculus:
  - differential calculus
  - integral calculus
- understand the limit process and that calculus centers around this concept

Prescribed Learning Outcomes

► **OVERVIEW AND HISTORY OF CALCULUS** *(Historical Development of Calculus)* It is expected that students will understand the historical background and problems that led to the development of calculus.	*It is expected that students will:* • describe the contributions made by various mathematicians and philosophers to the development of calculus, including: - Archimedes - Fermat - Descartes - Barrow - Newton - Leibniz - Jakob and Johann Bernoulli - Euler - L'Hospital
► **FUNCTIONS, GRAPHS, AND LIMITS** *(Functions and their Graphs)* It is expected that students will represent and analyze rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions, using technology as appropriate.	*It is expected that students will:* • model and apply inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions to solve problems • draw (using technology), sketch and analyze the graphs of rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions for: - domain and range - intercepts • recognize the relationship between a base a exponential function ($a > 0$) and the equivalent base e exponential function (convert $y = a^x$ to $y = e^{x(\ln a)}$) • determine, using the appropriate method (analytic or graphing utility) the points where $f(x) = 0$

Prescribed Learning Outcomes

► FUNCTIONS, GRAPHS, AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

It is expected that students will:

- demonstrate an understanding of the concept of limit and notation used in expressing the limit of a function as x approaches a: $\lim_{x \to a} f(x)$
- evaluate the limit of a function
  - analytically
  - graphically
  - numerically
- distinguish between the limit of a function $f(x)$ as x approaches a and the value of the function at $x = a$
- demonstrate an understanding of the concept of one-sided limits and evaluate one-sided limits
- determine limits that result in infinity (infinite limits)
- evaluate limits of functions as x approaches infinity (limits at infinity)
- determine vertical and horizontal asymptotes of a function, using limits
- determine whether a function is continuous at $x = a$

► THE DERIVATIVE *(Concept and Interpretations)*

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

It is expected that students will:

- describe geometrically a secant line and a tangent line for the graph of a function at $x = a$.
- define and evaluate the derivative at $x = a$ as: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
- define and calculate the derivative of a function using: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ or $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- use alternate notation interchangeably to express derivatives (i.e., $f'(x), \frac{dy}{dx}, y'$, etc.)
- compute derivatives using the definition of derivative
- distinguish between continuity and differentiability of a function at a point
- determine when a function is non-differentiable, and explain why
- determine the slope of a tangent line to a curve at a given point
- determine the equation of the tangent line to a curve at a given point
- for a displacement function $s = s(t)$, calculate the average velocity over a given time interval and the instantaneous velocity at a given time
- distinguish between average and instantaneous rate of change

Prescribed Learning Outcomes

► THE DERIVATIVE *(Computing Derivatives)*

It is expected that students will determine derivatives of functions using a variety of techniques.

It is expected that students will:

- compute and recall the derivatives of elementary functions including:
  - $\frac{d}{dx}(x^r)=rx^{r-1}$, r is real
  - $\frac{d}{dx}(e^x)=e^x$
  - $\frac{d}{dx}(\ln x)=\frac{1}{x}$
  - $\frac{d}{dx}(\cos x)=-\sin x$
  - $\frac{d}{dx}(\sin x)=\cos x$
  - $\frac{d}{dx}(\tan x)=\sec^2 x$
  - $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$
- use the following derivative formulas to compute derivatives for the corresponding types of functions:
  - constant times a function: $\frac{d}{dx}cu=c\frac{du}{dx}$
  - $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$ (sum rule)
  - $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ (product rule)
  - $\frac{d}{dx}(\frac{u}{v})=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ (quotient rule)
  - $\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$ (power rule)
- use the Chain Rule to compute the derivative of a composite function:

  $\frac{dy}{dx}=\frac{du}{dx}\frac{dy}{du}$ or $\frac{d}{dx}(F(g(x)))=g'(x)F'(g(x))$
- compute the derivative of an implicit function.
- use the technique of logarithmic differentiation.
- compute higher order derivatives

Prescribed Learning Outcomes

▶ APPLICATIONS OF DERIVATIVES *(Applied Problems)*

It is expected that students will solve applied problems from a variety of fields including the Physical and Biological Sciences, Economics, and Business.

It is expected that students will:

- solve problems involving displacement, velocity, acceleration
- solve related rates problems
- solve optimization problems (applied maximum / minimum problems)

▶ APPLICATIONS OF DERIVATIVES *(Derivatives and the Graph of the Function)*

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

It is expected that students will:

- given the graph of $y = f(x)$:
  - graph $y = f'(x)$ and $y = f''(x)$
  - relate the sign of the derivative on an interval to whether the function is increasing or decreasing over that interval.
  - relate the sign of the second derivative to the concavity of a function.
- determine the critical numbers and inflection points of a function.
- determine the maximum and minimum values of a function and use the first and / or second derivative test(s) to justify their solutions
- use Newton's iterative formula (with technology) to find the solution of given equations, $f(x) = 0$.
- use the tangent line approximation to estimate values of a function near a point and analyze the approximation using the second derivative.

Prescribed Learning Outcomes

► ANTIDIFFERENTIATION *(Recovering Functions from their Derivatives)*

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

It is expected that students will:

- explain the meaning of the phrase "$F(x)$ is an antiderivative (or indefinite integral) of f(x)."
- use antiderivative notation appropriately $\int f(x)dx$ (i.e., for the antiderivative of $f(x)$).
- compute the antiderivatives of linear combinations of functions whose individual antiderivatives are known including:
  - $\int k\,dx = kx + C$
  - $\int x^r dx = \frac{x^{r+1}}{r+1} + C$ if $r \neq -1$
  - $\int \frac{dx}{x} = \ln|x| + C$
  - $\int e^x dx = e^x + C$
  - $\int \sin x\,dx = -\cos x + C$
  - $\int \cos x\,dx = \sin x + C$
  - $\int \sec^2 x\,dx = \tan x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
  - $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- compute $\int f(ax+b)\,dx$ if $\int f(u)\,du$ is known.
- create integration formulas from the known differentiation formulas
- solve initial value problems using the concept that if $F'(x) = G'(x)$ on an interval, then $F(x)$ and $G(x)$ differ by a constant on that interval.

Prescribed Learning Outcomes

► **ANTIDIFFERENTIATION** *(Applications of Antidifferentiation)* It is expected that students will use antidifferentiation to solve a variety of problems.	*It is expected that students will:* • use antidifferentiation to solve problems about motion of a particle along a line that involve: - computing the displacement given initial position and velocity as a function of time - computing velocity and/or displacement given suitable initial conditions and acceleration as a function of time • use antidifferentiation to find the area under the curve $y = f(x)$, above the x-axis, from $x = a$ to $x = b$. • use differentiation to determine whether a given function or family of functions is a solution of a given differential equation • use correct notation and form when writing the general and particular solution for differential equations. • model and solve exponential growth and decay problems using a differential equation of the form: $\frac{dy}{dt} = ky$ • model and solve problems involving Newton's Law of Cooling using a differential equation of the form: $\frac{dy}{dt} = ay + b$.

Appendix B

Learning Resources

WHAT IS APPENDIX B?

Appendix B for this IRP includes Grade Collection resources for Grades 10 to 12 Mathematics courses selected in 1998. These resources are all provincially recommended. It is the minstry's intention to add additional resources to these Grade Collections as they are evaluated.
The titles are listed alphabetically and each resource is annotated. In addition, Appendix B contains information on selecting learning resources for the classroom.

What information does an annotation provide?

1. **General Description**

2. **Media Format**

3. **Author(s)**

4. **Cautions**

5. **Curriculum Organizers**

6. **Grade Level Grid**

Exploring Functions with the T1-82 Graphics Calculator

Author(s): *Kelly, B.*

General Description: This resource contains exercises and investigations that are suitable for a wide range of student abilities. It acts as a supplement to a limited number of learning outcomes and is designed for follow-up activities.

Caution:

Audience: *General*

Category: *Student, Teacher Resource*

Curriculum Organizer(s): *Patterns and Relations*

Recommended for:

	10	11	12
Principles of Mathematics	✓	✓	✓
Applications of Mathematics	✓	✓	
Essentials of Mathematics			

Supplier: *Pearson Education Canada*
26 Prince Andrew Place
Don Mills, ON
M3C 2T8
Tel: 1-800-361-6128 Fax: 1-800-563-9196

Price: Text: $14.60

ISBN/Order No: 1895997003

Copyright Year: 1993

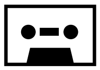
7. **Category**

8. **Audience**

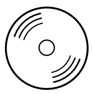
9. **Supplier**

1. ***General Description:*** This section provides an overview of the resource.

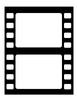
2. ***Media Format:*** This part is represented by an icon next to the title. Possible icons include:



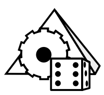
Audio Cassette



CD-ROM

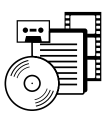


Film



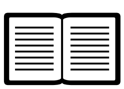
Games/Manipulatives

Laserdisc/Videodisc



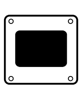
Multimedia

Music CD

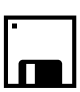


Print Materials

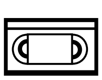
Record



Slides



Software



Video

3. ***Author(s):*** Author or editor information is provided where it might be of use to the teacher.

4. ***Cautions:*** This category is used to alert teachers about potentially sensitive issues.

5. ***Curriculum Organizers:*** This category helps teachers make links between the resource and the curriculum.

6. ***Grade Level Grid:*** This category indicates the suitable age range for the resource.

7. ***Category:*** This section indicates whether it is a student and teacher resource, teacher resource, or professional reference.

8. ***Audience:*** This category indicates the suitability of the resource for different types of students. Possible student audiences include the following:
   - general
   - English as a second language (ESL)
   - *Students who are:*
     - gifted
     - blind or have visual impairments
     - deaf or hard of hearing
   - *Students with:*
     - severe behavioural disorders
     - dependent handicaps
     - physical disabilities
     - autism
     - learning disabilities (LD)
     - mild intellectual disabilities (ID-mild)
     - moderate to severe/profound disabilities (ID-moderate to severe/profound)

9. ***Supplier:*** The name and address of the supplier are included in this category. Prices shown here are approximate and subject to change. Prices should be verified with the supplier.

What about the videos?

The ministry attempts to obtain rights for most provincially recommended videos. Negotiations for the most recently recommended videos may not be complete. For these titles, the original distributor is listed in this document, instead of British Columbia Learning Connection Inc. Rights for new listings take effect the year implementation begins. Please check with British Columbia Learning Connection Inc. before ordering new videos.

Selecting Learning Resources for the Classroom

Selecting a learning resource for Grades 10 to 12 Mathematics means choosing locally appropriate materials from the Grade Collection or other lists of evaluated resources. The process of selection involves many of the same considerations as the process of evaluation, though not to the same level of detail. Content, instructional design, technical design, and social considerations may be included in the decision-making process, along with a number of other criteria.

The selection of learning resources should be an ongoing process to ensure a constant flow of new materials into the classroom. It is most effective as an exercise in group decision making, co-ordinated at the school, district, and ministry levels. To function efficiently and realize the maximum benefit from finite resources, the process should operate in conjunction with an overall district and school learning resource implementation plan.

Teachers may choose to use provincially recommended resources to support provincial or locally developed curricula; choose resources that are not on the ministry's list; or choose to develop their own resources. Resources that are not on the provincially recommended list must be evaluated through a local, board-approved process.

Criteria for Selection

There are a number of factors to consider when selecting learning resources.

Content

The foremost consideration for selection is the curriculum to be taught. Prospective resources must adequately support the particular learning outcomes that the teacher wants to address. Teachers will determine whether a resource will effectively support any given learning outcomes within a curriculum organizer. This can only be done by examining descriptive information regarding that resource; acquiring additional information about the material from the supplier, published reviews, or colleagues; and by examining the resource first-hand.

Instructional Design

When selecting learning resources, teachers must keep in mind the individual learning styles and abilities of their students, as well as anticipate the students they may have in the future. Resources have been recommended to support a variety of special audiences, including gifted, learning disabled, mildly intellectually disabled, and ESL students. The suitability of a resource for any of these audiences has been noted in the resource annotation. The instructional design of a resource includes the organization and presentation techniques; the methods used to introduce, develop, and summarize concepts; and the vocabulary level. The suitability of all of these should be considered for the intended audience.

Teachers should also consider their own teaching styles and select resources that will complement them. Lists of provincially recommended resources contain materials that range from prescriptive or self-contained resources to open-ended resources that require considerable teacher preparation. There are provincially recommended materials for teachers with varying levels of experience with a particular subject, as well as those that strongly support particular teaching styles.

Technology Considerations

Teachers are encouraged to embrace a variety of educational technologies in their classrooms. To do so, they will need to ensure the availability of the necessary equipment and familiarize themselves with its operation. If the equipment is not currently available, then the need must be incorporated into the school or district technology plan.

Social Considerations

All resources on the ministry's provincially recommended lists have been thoroughly screened for social concerns from a provincial perspective. However, teachers must consider the appropriateness of any resource from the perspective of the local community.

Media

When selecting resources, teachers should consider the advantages of various media. Some topics may be best taught using a specific medium. For example, video may be the most appropriate medium when teaching a particular, observable skill, since it provides a visual model that can be played over and over or viewed in slow motion for detailed analysis. Video can also bring otherwise unavailable experiences into the classroom and reveal "unseen worlds" to students. Software may be particularly useful when students are expected to develop critical-thinking skills through the manipulation of a simulation, or where safety or repetition is a factor. Print resources or CD-ROM can best be used to provide extensive background information on a given topic. Once again, teachers must consider the needs of their individual students, some of whom may learn better from the use of one medium than another.

Funding

As part of the selection process, teachers should determine how much money is available to spend on learning resources. This requires an awareness of school and district policies, and procedures for learning resource funding. Teachers will need to know how funding is allocated in their district and how much is available for their needs. Learning resource selection should be viewed as an ongoing process that requires a determination of needs, as well as long-term planning to co-ordinate individual goals and local priorities.

Existing Materials

Prior to selecting and purchasing new learning resources, an inventory of those resources that are already available should be established through consultation with the school and district resource centres. In some districts, this can be facilitated through the use of district and school resource management and tracking systems. Such systems usually involve a database to help keep track of a multitude of titles. If such a system is available, then teachers can check the availability of a particular resource via a computer.

SELECTION TOOLS

The Ministry of Education has developed a variety of tools to assist teachers with the selection of learning resources.

These include:

- Integrated Resource Packages (IRPs) that contain curriculum information, teaching and assessment strategies, and provincially recommended learning resources, including Grade Collections
- resource databases on disks or on-line
- sets of the most recently recommended learning resources (provided each year to a number of host districts throughout the province to allow teachers to examine the materials first-hand at regional displays)
- sample sets of provincially recommended resources (available on loan to districts on request)

A MODEL SELECTION PROCESS

The following series of steps is one way a school resource committee might go about selecting learning resources:

1. Identify a resource co-ordinator (for example, a teacher-librarian).
2. Establish a learning resources committee made up of department heads or lead teachers.
3. Develop a school vision and approach to resource-based learning.
4. Identify existing learning resource and library materials, personnel, and infrastructure.
5. Identify the strengths and weaknesses of existing systems.
6. Examine the district Learning Resources Implementation Plan.
7. Identify resource priorities.
8. Apply criteria such as those found in *Evaluating, Selecting, and Managing Learning Resources: A Guide* to shortlist potential resources.
9. Examine shortlisted resources first-hand at a regional display or at a publishers' display, or borrow a set by contacting either a host district or the Curriculum Branch.
10. Make recommendations for purchase.

FURTHER INFORMATION

For further information on evaluation and selection processes, catalogues, CD-ROM catalogues, annotation sets, or resource databases, please contact the Curriculum Branch of the Ministry of Education.

INDUSTRY STANDARD SOFTWARE

It is expected that students in Grades 10 to 12 Mathematics will have access to grade-level appropriate productivity tools including spreadsheets, database packages, word processors, drawing and painting tools, and so on. Use of industry standard software is encouraged.

Reviews of appropriate software are regularly published in a variety of computer and trade magazines. Selection of a particular application should consider:

- existing hardware and upgrade path
- cross-platform capability
- instructor training requirements
- time spent on student skill development versus curricular intent
- cross-curriculum applicability
- general flexibility and utility

Software for Grades 10 to 12 Mathematics

Software that supports Grades 10 to 12 Mathematics includes but is not limited to:

Title	Function/Purpose
Excel	Spreadsheet
Lotus 123	Spreadsheet
Supercalc	Spreadsheet

Inclusion in this list does not constitute recommended status or endorsement of the product.

Appendix B

Learning Resources
Grade Collections

Calculus: A New Horizon, Sixth Edition

Author(s): *Anton, Howard*

General Description: Comprehensive resource package consisting of a student text and a student resource manual with sample tests covers all aspects of the Calculus 12 curriculum, with the exception of quadratic approximations. Text includes chapter and section overviews, section and chapter exercieses, answer key, and appendices. Student resource manual provides detailed solutions to odd-numbered exercises, and sample tests. A test bank, Student Resource and Survival CD, and an electronic version of the teacher's manual are optional resources.

Caution: *Software is not well implemented and does not add significantly to text. U.S. bias - little to no Canadian content. Exploration style questions weak. Little to no group work or exercises exercising student creativity.*

Audience: *General*

Category: *Student, Teacher Resource*

Curriculum Organizer(s): *Antidifferentiation*
Applications of Derivatives
The Derivative
Functions, Graphs and Limits
Overview and History of Calculus

Supplier: *John Wiley & Sons Canada Ltd.*
22 Worcester Road
Etobicoke, ON
M9W 1L1

Tel: 1-800-567-4797 Fax: 1-800-565-6802

Price: Textbook: $62.80
Student Resource Manual: $30.80

ISBN/Order No: Textbook: 0-471-15307-9
Student Resource Manual: 0-471-246239

Copyright Year: 1999

Calculus: Graphical, Numerical, Algebraic

Author(s): *Finney ... (et al.)*

General Description: This resource covers all topics in Calculus 12 with the exception of quadratic approximations of curves. The resource is well laid out and makes good use of technology to aid student learning. This resource makes a good attempt to allow for alternate teaching and learning styles through group work and journal writing assignments. Features include a balanced approach; applications to real-world problems; explorations that provide guided investigations; and exercise sets of various kinds. Selected answers and solutions are provided.

Audience: *General*

Category: *Student, Teacher Resource*

Curriculum Organizer(s): *Antidifferentiation*
Applications of Derivatives
The Derivative
Functions, Graphs and Limits
Overview and History of Calculus

Supplier: *Pearson Education Canada*
26 Prince Andrew Place
Don Mills, ON
M3C 2T8

Tel: 1-800-361-6128 Fax: 1-800-563-9196

Price: $99.25

ISBN/Order No: 0-201-32445-8

Copyright Year: 1999

Calculus of a Single Variable Early Transcendental Functions, Second Edition

Author(s): *Larson; Hostetler; Edwards*

General Description: Well-laid out, visually appealing resource that integrates the historical development of calculus, comprises nine chapters, appendices, solutions, and an index. The text encourages lecture or discovery approaches, pencil and paper skills or technology, and supports a variety of presentation styles. Concepts are clearly introduced and developed. Includes exercise sets, real-world explorations, and problems. An Interactive Calculus CD-ROM is available to support the text.

System Requirements for Windows: Windows '95; CD-ROM drive; 8Mb RAM ; 1Mb VRAM.

Caution: *Chapter motivators do include significant reference to commercial products. Teaching strategies are not laid out. Few references to First Nations people.*

Audience: *General*

Category: *Student, Teacher Resource*

Curriculum Organizer(s): *Applications of Derivatives*
The Derivative
Functions, Graphs and Limits
Overview and History of Calculus

Supplier: *Nelson Thomson Learning*
1120 Birchmount Road
Scarborough, ON
M1K 5G4

Tel: 1-800-268-2222 Fax: (416) 752-9365 or 752-9646

Price: $105.00

ISBN/Order No: 0-395-93321-8

Copyright Year: 1999

Calculus of a Single Variable, Sixth Edition

Author(s): *Larson; Hostetler; Edwards*

General Description: Resource package consisting of a student text, instructor's resource guide, two-volume solutions guide, and optional Interactive Calculus CD-ROM integrates technology and real-world problems in a ten chapter format. The readable text provides motivational problems and explanations, and well-written examples take a variety of solution approaches, follow clearly defined strategies, and build in checking for reasonability of the solution. Exercises vary in level of difficulty and incorporate writing and thinking activities. The instructor's resource guide provides teaching strategies, tests and answer keys, and the complete solutions guides contain detailed exercise solutions. An optional Interactive Calculus CD-ROM provides an electronic version of the text.

Caution: *The CD-ROM package does not cover the whole curriculum and is only an electronic version of the text.*

Audience: *General*

Category: *Student, Teacher Resource*

Curriculum Organizer(s): *Antidifferentiation*
Applications of Derivatives
The Derivative
Functions, Graphs and Limits
Overview and History of Calculus

Supplier: *Nelson Thomson Learning*
1120 Birchmount Road
Scarborough, ON
M1K 5G4

Tel: 1-800-268-2222 Fax: (416) 752-9365 or 752-9646

Price: Textbook: $72.50
Solutions Guide Volume 1: $27.00
Solutions Guide Volume 2: $27.00
Instructor's Resource Guide: $15.50

ISBN/Order No: Textbook: 0-395-88578-7
Solutions Guide Volume 1: 0-395-88769-0
Solutions Guide Volume 2: 0-395-88770-4
Instructor's Resource Guide: 0-395-88766-6

Copyright Year: 1998

Single Variable Calculus Early Transcendentals, Fourth Edition

Author(s): *Stewart, James*

General Description: Comprehensive resource package consisting of a student text with demo CD-ROM learning tool, student solutions manual, and a study guide focuses on conceptual understanding of problems with an application to real-world situations. Text includes explanations and examples of the conceptual ideas, projects, graded exercises, answer key, and a demo CD-ROM that presents an interactive challenge to students. The student solutions manual presents problem solving strategies, and completely worked-out solutions. The study guide offers additional explanations and worked-out examples.

System Requirements for Windows: Windows '95; CD-ROM drive; 8Mb RAM ; 1Mb VRAM.

Audience: *General*

Category: *Student, Teacher Resource*

Curriculum Organizer(s): *Antidifferentiation*
Applications of Derivatives
The Derivative
Functions, Graphs and Limits
Overview and History of Calculus

Supplier: *Nelson Thomson Learning*
1120 Birchmount Road
Scarborough, ON
M1K 5G4

Tel: 1-800-268-2222 Fax: (416) 752-9365 or 752-9646

Price: Textbook: $68.00
Study Guide: $29.00
Student Solutions Manual: $32.00

ISBN/Order No: Textbook: 0-534-35563-3
Study Guide: 0-534-36820-4
Student Solutions Manaul: 0-534-36301-6

Copyright Year: 1999

▼ SAMPLE 10: CALCULUS 12

Topic: *Limits*

Prescribed Learning Outcomes:

Problem Solving

It is expected that students will:

- analyse a problem and identify the significant elements
- demonstrate the ability to work individually and co-operatively to solve problems
- determine that their solutions are correct and reasonable
- clearly communicate a solution to problem and justify the process used to solve it
- use appropriate technology to assist in problem solving

Functions, Graphs and Limits (Limits)

It is expected that students will:

- demonstrate an understanding of the concept of limit and notation used in expressing the limit of a function $f(x)$ as x approaches a: $\lim_{x \to a} f(x)$
- evaluate the limit of a function:
  - analytically
  - graphically
  - numerically
- distinguish between the limit of a function as x approaches a and the value of the function at $x = a$
- demonstrate an understanding of the concept of one-sided limits and evaluate one-sided limits
- determine limits that result in infinity (infinite limits)
- evaluate limits of functions as x approaches infinity (limits at infinity)
- determine vertical and horizontal asymptotes of a function, using limits
- determine whether a function is continuous at $x = a$

In addition to these outcomes, the teacher assessed outcomes related to the historical development of calculus, students' attitudes, and group and communication skills.

UNIT FOCUS

The concept of a limit plays a central role in students future understanding of the importance and usefulness calculus. It is this understanding that enables students to effectively interpret the results of a variety of problem solutions using calculus.

PLANNING THE UNIT

To plan the unit, the teacher:

- identified the prescribed learning outcomes for instruction and assessment for the unit and the prerequisite knowledge and skills needed to achieve these outcomes.
- determined which of these prerequisites were already in place and which to review
- looked for ways to connect students' learning to other desirable learning outcomes, including those associated with group-work skills, communication skills, and attitudes
- identified the criteria to use to evaluate students' success in the unit
- designed assessment to be an integral part of the instructional process
- began the unit by reviewing and discussing with students characteristics of functions that they were familiar with (exponential, logarithmic, sinusoidal, etc.)

THE UNIT

Introducing the Concept of Limit

- The unit began with a review of the characteristics of familiar functions and their graphs.
- The teacher worked with the students to develop the concept of limit using an overhead graphing utility to demonstrate graphically the relationship between a variety of functions and the limits of the function at a number of different points.
- Students used graphing calculators to perform "what if" investigations to help them determine graphically the effect on the limits of different types of functions if the characteristic equation of the function were altered.

One-sided Limits

- Students were encouraged to explore the concept of one-sided limits by comparing functions with one-sided limits to those that had different left-hand and right-hand limits.

Infinite Limits and Limits at Infinity

- The teacher provided examples illustrating how to calculate limits of functions that result in infinity and limits of functions as x approaches infinity.
- Guided practice reinforced students' understanding of the difference between infinite limits and limits at infinity.

Asymptotes of a Function

- The teacher then guided class discussion from infinite limits and limits at infinity to vertical and horizontal asymptotes of a function
- Students were given practice determining the vertical and horizontal asymptotes of different functions, using limits.

Continuous Functions

- The teacher began by demonstrating that a continuous function is one which can be graphed over each interval of its domain with one continuous motion of the pen.
- This concept was expanded to the use of a graphing calculator where graphs of functions that were not continuous at $x = a$ had "holes" were no pixels were represented.
- Students were then asked to use limits to describe this phenomenon. The teacher reinforced that a function is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$

Applying Knowledge of Limits

To encourage students to apply their knowledge of limits to solve some of the historical problems that the teacher had introduced at the beginning of the unit (i.e., tangent line problem and the area problem). Students worked in small groups to develop possible solutions to the problems.

DEFINING THE CRITERIA

Mathematical Thinking

to what extend do students:

- demonstrate an understanding of the concept of limit
- can evaluate the limit of a function in a variety of ways and show the connection between the methods (analytical, graphical, numerical)
- describe why limits are important and how they can be used
- describe the difference between infinite limits and limits at infinity

- use limits to determine asymptotes of a function and to determine whether the function is continuous at a given point.

Attitudes

To what extent do students:

- approach problem situations with confidence
- show a willingness to persevere in solving difficult problems
- show flexibility in using available resources (e.g., graphing calculators, textbooks, help from other students or the teacher)

Group Skills

To what extent do students:

- work with other students in whole-class discussions and small groups, to build on ideas and understanding
- initiate, develop, and maintain interactions within the group
- help other students develop understanding

Communication Skills

To what extend do students:

- communicate ideas clearly and understandably
- listen to and make use of the ideas of other students

ASSESSING AND EVALUATING STUDENT PERFORMANCE

Observation and Questioning

Students' understanding, attitudes, and group and communication skills were evaluated informally throughout the unit. The pace of the unit was determined in part by the speed with which students seemed to grasp the concepts.

The teacher:

- observed students as they participated in whole-class and small-group activities looking for evidence that they understood the concepts, built on ideas of others, initiated, developed, and maintained interactions within the group, and assisted others in developing understanding
- reviewed students' work to note the extent to which they met the criteria for mathematical thinking
- made note of behaviours that students were or were not achieving the criteria established for the unit (e.g., using graphing calculators to determine limits when appropriate)
- used questions to assess students' understanding of central concepts, the level of confidence they displayed when problem solving, and their willingness to persevere when solving difficult problems
- checked students' work to note the extent to which they used appropriate limit terminology and notation.

Individual Projects

Each student completed a research project on an topic or mathematician of personal interest. The project required students to:
- use what they learned about limits
- describe the topic and why it was of interest
- describe how the use of limits related to their topic
- explain what their finding meant, why they were relevant, and why they were organized as they were
- use and cite a variety of appropriate information sources
- complete a written report, and present their findings to the class

Presentations and reports were evaluated separately using the following holistic scale. Students received copies of the scale before starting the project and used it to rate their presentations and written reports. The teacher held conferences to discuss discrepancies between students' and teacher's ratings. Students were given suggestions for improvements. Students who received scores of one or two on their written reports were given the opportunity to redo them. The final score for the written report was the higher of the two scores.

Project Rating Scale

4 - Outstanding	Information is presented clearly, logically, and understandably. Examples or demonstrations are used appropriately to illustrate explanations. Findings are well organized and effectively displayed. Explanations indicate a clear understanding of the topic and the use of limits in a variety of contexts (e.g., to determine asymptotes of a function). References are appropriate for the topic and indicate that the student understands where to look for information.
3 - Adequate	The presentation indicates that the student has a basic understanding of the topic and of limits. Information is understandable. Findings are organized and displayed acceptably. References are appropriate for the Topic.
2 - Needs Improvement	The presentation indicates a limited understanding of either the topic or the use of limits, or both. The presentation may be illogical or difficult to follow. Findings may be organized poorly or ineffectively. References may indicate that the student is not clear on finding the best sources of information.
1 - Inadequate	The presentation indicates a lack of understanding of either the topic or the use of limits. The presentation is illogical and difficult to follow. Findings are poorly organized and ineffectively presented. References may be lacking or inappropriate for the topic.

Test/Team Competition

To evaluate students' understanding of the concepts developed in this unit, the teacher designed a team competition based on completing a set of items on a study sheet. Students worked in groups of four and used all available resources to answer each of the items. The groups were given opportunities to drill each other. When all members of the group felt comfortable with the information, they were asked to take a test individually that contained items similar to those on the study sheet. The teacher collected the tests, scored them, and gave each student a grade. Groups worked together to correct their tests and returned their corrections for additional points. The winning group had the highest average individual test scores and the highest average scores on their corrections.

Take-Home Test

Students were given a take-home test that contained similar problems to those studied during the unit. As part of the test, students were required to explain the rules they had learned for working with limits and use the rules to solve examples that illustrated their explanations.

The last page of the test included a self-evaluation sheet that asked the following questions:

- What things in this unit did your find easy?
- What things did you find difficult?
- Are there any areas in the unit that you need more help with? If so, what are they?
- Would you be interested in after-school peer tutoring for help in these areas?
- Would you be willing to tutor another student after school in the areas you feel most comfortable with?

The teacher used the results of the take-home test and students' responses to the self-evaluation to pair students for after-school peer tutoring. Students were required to correct errors on their take-home tests and resubmit them for a second evaluations.

Appendix G

Illustrative Examples

Calculus 12

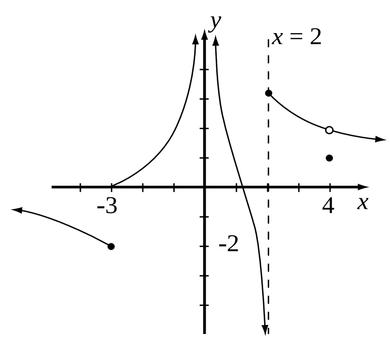
FUNCTIONS, GRAPHS AND LIMITS *(Functions and their Graphs)*

It is expected that students will represent and analyse inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions, using technology as appropriate.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will:	
• model and apply inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions to solve problems	c Find the area of the part of the first quadrant that is inside the circle $x^2 + y^2 = 4$ and to the left of the line $x = u$. (The inverse sine function will be useful here.) c Suppose that under continuous compounding 1 dollar grows after t years to $(1.05)^t$ dollars. Find the number r such that $(1.05)^t = e^{rt}$. (This r is called the nominal yearly interest rate.)
• draw (using technology), sketch and analyse the graphs for rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions, for: - domain and range - intercepts	c Sketch the graph of $\frac{x^2}{x^2-1}$. c Write $\sin(\tan^{-1} x)$ in a form that does not involve any trigonometric functions. c a) Let $f(x) = \ln(ex)$. Sketch the graph of $f(x)$. b) Let $g(x)$ be the inverse function of $f(x)$. Sketch the graph of $g(x)$. c) Find an explicit formula for $g(x)$. c Sketch the curve $y = \ln(e^x + 1)$.
• recognize the relationship between a base a exponential function (a>0) and the equivalent base e exponential function (convert $y = a^x$ to $y = e^{x(\ln a)}$) (Differentiation problems will also require students to demonstrate achievement of this outcome.)	c Let $f(x) = x^x$. a) Express $f(x)$ in the form $f(x) = e^{g(x)}$. b) Find the minimum value taken on by $f(x)$ on the interval $(0, \infty)$.
• determine using the appropriate method (analylitic or graaphing utility) the points where $f(x) = 0$	c [No example for this prescribed learning outcome}

FUNCTIONS, GRAPHS AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • demonstrate an understanding of the concept of limit and notation used in expressing the limit of a function $f(x)$ as x approaches a: $\lim\limits_{x \to a} f(x)$	c * Sketch and explain what each of the following means: a) $\lim\limits_{x\to a} f(x) = L$ b) $\lim\limits_{x\to a^-} f(x) = L$ c) $\lim\limits_{x\to a^+} f(x) = L$ d) $\lim\limits_{x\to a^+} f(x) = \infty$ e) $\lim\limits_{x\to a^-} f(x) = \infty$ f) $\lim\limits_{x\to \infty} f(x) = L$
• evaluate the limit of a function - analytically - graphically - numerically • distinguish between the limit of a function as x approaches a and the value of the function at $x = a$ • demonstrate an understanding of the concept of one-sided limits and evaluate one-sided limits • determine limits that result in infinity (infinite limits) • evaluate limits of functions as x approaches infinity (limits at infinity)	c Use a calculator to draw conclusions about: a) $\lim\limits_{t\to 0} \frac{\sin t - t}{t^3}$ b) $\lim\limits_{x\to 0^+} x^x$  c From the given graph find the following limits. If there is no limit, explain why: a) $\lim\limits_{x\to 2^+} f(x)$ b) $\lim\limits_{x\to 2^-} f(x)$ c) $\lim\limits_{x\to -3^+} f(x)$ d) $\lim\limits_{x\to -3^-} f(x)$ e) $\lim\limits_{x\to -3} f(x)$

FUNCTIONS, GRAPHS AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

Prescribed Learning Outcomes	Illustrative Examples
	f) $\lim_{x \to 4} f(x)$ g) $f(4)$ h) $\lim_{x \to 0} f(x)$ C Find the limits: a) $\lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$ b) $\lim_{t \to 6} \frac{17}{(t-6)^2}$ c) $\lim_{h \to 0} \frac{(2+h)^{-2} - \frac{1}{4}}{h}$ d) $\lim_{v \to 2} \frac{v^2 + 2v - 8}{v^4 - 16}$ e) $\lim_{x \to 8^-} \frac{\lvert x-8 \rvert}{x-8}$ f) $\lim_{x \to 0} \frac{1 - \sqrt{1-x^2}}{x}$ g) $\lim_{x \to 0} \frac{\sin 3x}{x}$ h) $\lim_{\theta \to 0} \frac{\tan 3\theta}{\theta}$ i) $\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin 5\theta}$ j) $\lim_{x \to \infty} \frac{3x^2 + 5}{4 - x^2}$ k) $\lim_{x \to \infty} x \sin \frac{1}{x}$ l) $\lim_{x \to \infty} \frac{\sqrt{x^2 + 4x}}{4x + 1}$ m) $\lim_{x \to 0^+} \frac{1}{2 + 10^{\frac{1}{x}}}$ n) $\lim_{x \to 0^-} \frac{1}{2 + 10^{\frac{1}{x}}}$

FUNCTIONS, GRAPHS AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

Prescribed Learning Outcomes	Illustrative Examples
• determine vertical and horizontal asymptotes of a function, using limits	c Find the horizontal and vertical asymptotes of each of the following curves: a) $y=\dfrac{x}{x+4}$ b) $y=\dfrac{x^2}{x^2-1}$ c) $y=\dfrac{x^2}{x^2+1}$ d) $y=\dfrac{1}{(x-1)^2}$
• determine whether a function is continuous at $x=a$	c *Given $f(x)=\begin{cases}\sqrt{-x} & ; \quad x<0 \\ 3-x & ; \quad 0\le x\le 3 \\ (x-3)^2 & ; \quad x>3\end{cases}$ a) find: i) $\lim\limits_{x\to 0^+} f(x)$ ii) $\lim\limits_{x\to 0^-} f(x)$ iii) $\lim\limits_{x\to 0} f(x)$ iv) $\lim\limits_{x\to 3^-} f(x)$ v) $\lim\limits_{x\to 3^+} f(x)$ vi) $\lim\limits_{x\to 3} f(x)$ b) Where is the graph discontinuous. Why? c) Sketch the graph.

THE DERIVATIVE *(Concept and Interpretations)*

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • describe geometrically a secant line and a tangent line for the graph of a function at $x = a$. • define and evaluate the derivative at $x = a$ as: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ • define and calculate the derivative of a function using: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ or $f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ • use alternate notation interchangeably to express derivatives (i.e., $f'(x), \frac{dy}{dx}, y'$, etc.) • compute derivatives using the definition of derivative	c Calculate the derivatives of the given function directly from the definition: a) $f(x) = x^2 + 3x$ b) $g(x) = \frac{x}{x-1}$ c) $H(t) = \sqrt{t+1}$ c Using the definition of derivative, find the equation of the tangent line to the curve: $y = 5 + 4x - x^2$ at the point $(2,9)$.
• distinguish between continuity and differentiability of a function at a point • determine when a function is non-differentiable, and explain why • determine the slope of a tangent line to a curve at a given point • determine the equation of the tangent line to a curve at a given point	c Determine where $f(x) = \|x-3\|$ is non differentiable. c Find an equation of the line tangent to the given curve at the point indicated. Sketch the curve: $y = \frac{1}{x^2+1}$ at $x = -1$ c *Find all the values of a for which the tangent line to $y = \ln x$ at $x = a$ is parallel to the tangent line to $y = \tan^{-1} x$ at $x = a$.

THE DERIVATIVE *(Concept and Interpretations)*

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

Prescribed Learning Outcomes	Illustrative Examples
• for a displacement function $s = s(t)$, calculate the average velocity over a given time interval and the instantaneous velocity at a given time • distinguish between average and instantaneous rate of change	c The position of a particle is given by $s = t^3$. Find the velocity when $t = 2$. c *The displacement in metres of a particle moving in a straight line is given by $s = 5 + 4t - t^2$ where t is measured in seconds. a) Find the average velocity from $t = 2$ to $t = 3$, $t = 2$ to $t = 2.1$, and $t = 2$ to $t = 2.01$. b) Find the instantaneous velocity when $t = 2$. c) Draw the graph of s as a function of t and draw the secant lines whose slopes are average velocity as in a). d) Draw the tangent line whose slope is the instantaneous velocity in b).

THE DERIVATIVE *(Computing Derivatives)*

It is expected that students will determine derivatives of functions using a variety of techniques.

Prescribed Learning Outcomes	Illustrative Examples

It is expected that students will:

- compute and recall the derivatives of elementary functions including:
  - $\frac{d}{dx}(x^r) = rx^{r-1}$, r e $\Re$
  - $\frac{d}{dx}(e^x) = e^x$
  - $\frac{d}{dx}(\ln x) = \frac{1}{x}$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- use the following derivative formulas to compute derivatives for the corresponding types of functions:
  - constant times a function: $\frac{d}{dx}cu = c\frac{du}{dx}$
  - $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ (sum rule)
  - $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (product rule)
  - $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (quotient rule)
  - $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$ (power rule)
- use the Chain Rule to compute the derivative of a composite function:
  $\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$ or $\frac{d}{dx}(F(g(x))) = g'(x)F'(g(x))$

c Calculate the derivatives of the following functions:

a) $y = 3 - 4x - 5x^2 + 6x^2$

b) $z = \frac{S^5 - S^3}{15}$

c) $f(t) = \frac{4}{2-5t}$

d) $y = (2x+3)^6$

e) $y = (x^2+9)\sqrt{x^2+3}$

f) $f(x) = \sin 2x$

g) $y(x) = \cos^2(5-4x^3)$

h) $y = \ln(3t^2+6)$

i) $y = 2e^{-x}$

j) $y = \tan(e^x)$

k) $F(x) = \log_5(3x - 8)$

l) $y = 2^{x^4}$

m) $y = \sin^{-1}(\sqrt{2x})$

n) $y = \tan^{-1}(3x)$

c Let $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$. Find $f'(x)$ and simplify.

c A certain function $f(x)$ has $f(1) = 4$ and $f'(1) = 5$. Let $g(x) = \frac{1}{\sqrt{2f(x)+1}}$. Find $g'(1)$.

THE DERIVATIVE *(Computing Derivatives)*

It is expected that students will determine derivatives of functions using a variety of techniques.

Prescribed Learning Outcomes	Illustrative Examples
• compute the derivative of an implicit function.	c Find $\dfrac{dy}{dx}$ in terms of x and y if: a) $xy = x + 2y + 1$ b) $x^3y + xy^5 = 2$ c Find the equation of the tangent line at the given point: $x^2y^3 - x^3y^2 = 12$ at $(-1,2)$ c Find y'' by implicit differentiation: $x^2 - y^2 = 1$ c The equation $x^4 + x^2y + y^4 = 27 - 6y$ defines y implicitly as a function of x near the point (2,1). Calculate y’ and y” at (2,1).
• use the technique of logarithmic differentiation.	c Let $y = (x+1)^2(2x+1)^3(3x+1)^4e^{5x}$. Use logarithmic differentiation to find $\dfrac{dy}{dx}$. c Using logarithmic differentiation, find a) y' if $y = x^x (x > 0)$ b) $\dfrac{dy}{dx}$ if $y = 5^{2x+1}$ c) $f'(x)$ if $f(x) = \dfrac{x^5\sqrt{3+x^2}}{(4+x^2)^3}$
• compute higher order derivatives	c Find y', y'', and y''' for the functions: a) $y = x^2 - \dfrac{1}{x}$ b) $y = \dfrac{x-1}{x+1}$ c Let $f(x) = \dfrac{1}{1+x}$. Find $f^{(7)}(x)$, the seventh derivative of $f(x)$.

APPLICATIONS OF DERIVATIVE *(Derivatives and the Graph of the Function)*

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • given the graph of $y = f(x)$: - graph $y = f'(x)$ and $y = f''(x)$ - relate the sign of the derivative on an interval to whether the function is increasing or decreasing over that interval. - relate the sign of the second derivative to the concavity of a function • determine the critical numbers and inflection points of a function. • determine the maximum and minimum values of a function and use the first and / or second derivative test(s) to justify their solutions	c For the following functions, find the critical numbers, the inflection points, and the vertical and horizontal asymptotes. Sketch the graph, and verify using a graphing calculator. a) $f(x) = \frac{x^3}{x^2 - 1}$ b) $y = \frac{3x^2}{2x^2 + 1}$ c) $f(x) = \frac{1}{x^2 - x}$ d) $f(x) = (x^2 - 1)^{\frac{2}{3}}$ c Find the critical and inflection points for $f(x) = -2xe^{-x}$ and sketch the graph of $f(x)$ for $x \geq 0$. c Sketch the graph $g(x) = x + \cos x$. Determine where the function is increasing most rapidly and least rapidly.
• use Newton's iterative formula (with technology) to find the solution of given equations, $f(x) = 0$	c Using a calculator, complete at least 5 iterations of Newton's method for $g(x) = x - \sin x + 1$ if $x_0 = -1$. c Explain why the function *f*, where $f(x) = \frac{2\ln x}{1 + x^2}$, has exactly one critical number. Use Newton's Method to find that critical number, correct to two decimal places. c Use Newton's Method to find, correct to two decimal places, the *x*-coordinate of a point at which the curve $y = \tan x$ meets the curve $y = x + 2$. Also look at the problem by using the "solve" feature of your calculator and by graphing the curves.

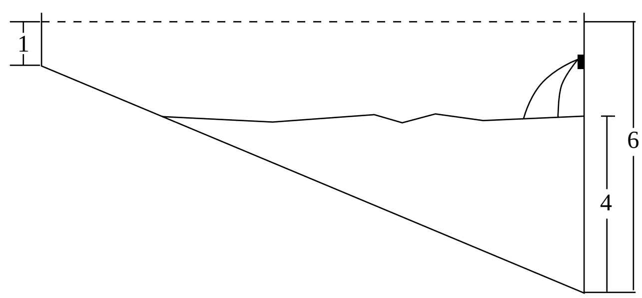
APPLICATIONS OF DERIVATIVE *(Derivatives and the Graph of the Function)*

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

Prescribed Learning Outcomes	Illustrative Examples
• use the tangent line approximation to estimate values of a function near a point and analyse the approximation using the second derivative.	c Determine the tangent line approximation for $f(x) = \sin x$ in the neighbourhood of $x = 0$. Zoom in on both graphs and compare the results. c A certain curve has equation $e^{2y} + y = x^2$. Note that the point (1,0) lies on the curve. Use a suitable tangent line approximation to give an estimate of the y-coordinate of the point on the curve that has x-coordinate equal to 1.2. c A certain function $f(x)$ has derivative given by $f'(x) = \sqrt[3]{x^2 - 1}$. It is also known that $f(3) = 4$. a) Use the tangent line approximation to approximate $f(3.2)$. b) Use the second derivative of f to determine whether the approximation in part (a) is bigger or smaller than the true value of $f(3.2)$. c The diameter of a ball bearing is found to be 0.48 cm, with possible error ± 0.005 cm. Use the tangent line approximation to approximate: a) the largest possible error in computing the volume of the ball bearing, b) the maximum percentage error

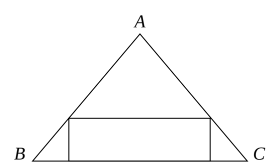
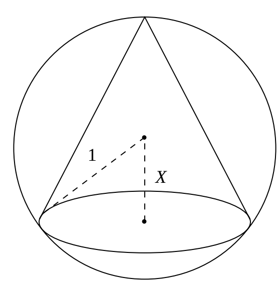
APPLICATIONS OF DERIVATIVE *(Applied Problems)*

It is expected that students will solve applied problems from a variety of fields including the physical and biological sciences, economics, and business.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will:	
• solve problems involving distance, velocity, acceleration	c The average velocity of a particle in the time interval from 1 to t is $\frac{e^t}{e^t - 1}$ for $t > 1$. Find the displacement and velocity at time $t = 2$.
• solve related rate problems (rates of change of two or more related variables that are changing with respect to time)	c Car P (a police car) was travelling in a northerly direction along the y-axis at a steady 50 kilometres per hour, while car Q was travelling eastward along the x-axis at varying speeds. At the instant when P was 60 metres north of the origin, Q was 90 metrs east of the origin. A radar unit on P recorded that the straight-line distance between P and Q was, at that instant, increasing at the rate of 80 kilometres per hour. How fast was car Q going? c *A swimming pool is 25 metres wide and 25 metres long. When the pool is full, the water at the shallow end is 1 metre deep, and the water at the other end is 6 metres deep. The bottom of the pool slopes at a constant angle from the shallow end to the deep end. Water is flowing into the deep end of the pool at 1 cubic meter per minute. How fast is the depth of water at the deep end increasing when the water there is 4 metres deep?  c *Boyle's Law* states that if gas is compressed but kept at constant temperature, then the pressure P and the volume V of the gas are related by the equation $PV=C$, where C is a constant that depends on temperature and the quantity of gas present. At a certain instant, the volume of gas inside a pressure chamber is 2000 cubic centimetres, the pressure is 100 kilopascals, and the pressure is increasing at 15 kilopascals per minute. How fast is the volume of the gas decreasing at this instant?

APPLICATIONS OF DERIVATIVE *(Applied Problems)*

It is expected that students will solve applied problems from a variety of fields including the physical and biological sciences, economics, and business.

Prescribed Learning Outcomes	Illustrative Examples
• solve optimization problems (applied maximum/minimum problems)	c A striped ball is thrown vertically upward, having a velocity of 12 m/s. After t seconds its altitude is represented by $s = 12t - 4.9t^2$. At what instant will it reach a maximum height and how high will it rise? c *Construct an open-topped box with a maximum volume from a 20 cm x 20 cm piece of cardboard by cutting out four equal corners. Use more than one method. Compare your results. c Triangle ABC has AB=AC=8, BC=10. Find the dimensions of the rectangle of maximum area that can be inscribed in ΔABC with one side on the rectangle on BC.  c A pharmaceuticals company can sell its veterinary antibiotic for a price of \$450 per unit. Assume that the total cost of producing x units in a year is given by $C(x)$ where $C(x) = 800\,000 + 50x + 0.004x^2$. How many units per year should the company produce in order to maximize yearly profit? c Find the volume of the cone of maximum volume that can be inscribed in a sphere of radius 1. Hint: let X be as shown in the diagram. 

ANTIDIFFERENTIATION *(Recovering Functions from their Derivatives)*

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • explain the meaning of the phrase "$F(x)$ is an antiderivative (or indefinite integral) of $f(x)$" • use antiderivative notation appropriately (i.e., $\int f(x)dx$ for the antiderivative of $f(x)$) • compute the antiderivatives of linear combinations of functions whose individual antiderivatives are known including: - $\int k\,dx = kx + C$ - $\int x^r dx = \frac{x^{r+1}}{r+1} + C$ if $r \neq -1$ - $\int \frac{dx}{x} = \ln\lvert x\rvert + C$ - $\int e^x dx = e^x + C$ - $\int \sin x\,dx = -\cos x + C$ - $\int \cos x\,dx = \sin x + C$ - $\int \sec^2 x\,dx = \tan x + C$ - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ - $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ • compute $\int f(ax+b)\,dx$ if $\int f(u)\,du$ is known	c Is it true that $\int 2xe^{2x}dx = xe^{2x} - \frac{e^{2x}}{2} + C$? c Is it true that $\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$? c Is it true that $\int x\cos x\,dx = \frac{x^2}{2}\sin x + C$? c Evaluate the following indefinite integrals: a) $\int (1+3x)^5 dx$ b) $\int e^{x/2}\,dx$ c) $\int \frac{5dx}{(1+4x)^2}$ c Suppose that $\frac{d^2y}{dx^2} = \sin\pi x$. a) Find a general formula for $\frac{dy}{dx}$ as a function of x. b) Find a general formula for y as a function of x.
• create integration formulas from the known differentiation formulas	c Let a be a positive constant other than 1. Find $\int a^x dx$. c Use the fact that $\sec^2 x = 1 + \tan^2 x$ to find $\int \tan^2 x\,dx$. c Use the identity $\cos 2x = 1 - 2\sin^2 x$ to find $\int \sin^2 x\,dx$. c Find $\int x\cos 3x\,dx$ by guessing that one antiderivative is $x\sin 3x$, differentiating, and then adjusting your guess.

ANTIDIFFERENTIATION *(Recovering Functions from their Derivatives)*

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

Prescribed Learning Outcomes	Illustrative Examples
• solve initial value problems using the concept that if $F'(x) = G'(x)$ on an interval, then $F(x)$ and $G(x)$ differ by a constant on that interval	c Suppose that $f''(t) = 3t^2$ for all t and $f(1) = 2$, $f'(1) = 5$. Find a formula for $f(t)$. c It is known that $f'(x) = e^{x/2}$ and $f(6 \ln 2) = 10$. Find a general formula for $f(x)$. c a) Verify that the function f given by $f(x) = 3e^{-x} \sin 3x - e^{-x} \cos 3x$ is an antiderivative of $10e^{-x} \cos 3x$. b) Use the result of part a) to find an antiderivative $G(x)$ of $10e^{-x} \cos 3x$ such that $G(0) = 5$.

Antidifferentiation *(Applications of Antidifferentiation)*

It is expected that students will use antidifferentiation to solve a variety of problems.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • use antidifferentiation to solve problems about motion along a line that involve: - computing the displacement given initial position and velocity as a function of time - computing velocity and/or displacement given suitable initial conditions and acceleration as a function of time • use antidifferentiation to find the area under the curve $y = f(x)$, above the x-axis, from $x = a$ to $x = b$	c A particle is moving back and forth along the x-axis, with velocity at time t given by $v(t) = \sin(t/2)$. When $t = 0$, the particle is at the point with coordinates (4,0). Where is the particle at time $t = \pi$? c Let a be a given positive number. Find the area of the region which is under the curve with equation $y = \frac{1}{x}$, above the x-axis, and between the vertical lines $x = a$ and $x = 2a$. Simplify your answer. c Use calculus to find the area under the curve, above the x-axis, and lying between the lines $x = \frac{1}{\sqrt{3}}$ and $x = \sqrt{3}$.
• use differentiation to determine whether a given function or family of functions is a solution of a given differential equation • use correct notation and form when writing the general and particular solution for differential equations • model and solve exponential growth and decay problems using a differential equation of the form: $\frac{dy}{dt} = ky$	c a) Verify that $y = \sin 3t$ is a solution of the differential equation $y'' = -9y$. b) Find a solution of the above differential equation that is not a constant multiple of $\sin 3t$. c) Find a solution y of the differential equation such that $y(0) = 2$ and $y'(0) = 1$. c *Torricelli's Law says that if a tank has liquid in it to a depth h, and there is a hole in the bottom of the tank, then liquid leaves the tank with a speed of $\sqrt{2gh}$. Use the metre as the unit of length and measure time in seconds. Then g is about 9.81. Take in particular a cylindrical tank with base radius R, with a hole of radius r at the bottom. a) Verify that h satisfies the differential equation $\frac{dh}{dt} = \frac{r^2}{R^2}\sqrt{2gh}$ b) Verify that for any constant C, the function $h(t)$ is a solution of the above equation, where $h(t) = \frac{1}{4}\left(C + \frac{\sqrt{2g}\,r^2}{R^2}t^2\right)^2$ c) A hot water tank has a base radius of 0.3 metres and is being drained through a circular hole of radius 0.1 at the bottom of the tank. At a certain time, the depth of the water in the tank is 1.5 metres. What is the depth of the water 30 seconds later?

ANTIDIFFERENTIATION (*Applications of Antidifferentiation*)

It is expected that students will use antidifferentiation to solve a variety of problems.

Prescribed Learning Outcomes	Illustrative Examples
	c *Shortly after a person was exposed to iodine-131, the level of radioactivity in the person's thyroid was 20 times the "safe" level. Three hours later, the level had decreased to 15.5 time the safe level. How much longer must the person wait until the level of radioactivity in the thyroid reaches "safe" territory? c *A simple model for the growth of a fish of a particular species goes as follows: Let M be the maximum length that can be reached by a member of that species. At the instant when the length of an individual of that species is y, the length of the individual is growing at a rate proportional to $M-y$, that is, y satisfies the differential equation $\frac{dy}{dt}=k(M-y)$ for some constant k. a) Let $w=M-y$. Verify that w satisfies the differential equation $\frac{dw}{dt}=-kw$. b) Write down the general solution of the differential equation of part a). Then write down a general formula for y as a function of time. c) Suppose that for a certain species, the maximum achievable length M is 60 cm, and that when t is measured in years, k = 0.05. A fish of that species has current length of 10 cm. How long will it take to reach a length of 20c m?
• model and solve problems involving Newton's Law of Cooling using a differential equation of the form: $\frac{dy}{dt}=ay+b$	c *Coffee in a well-insulated cup started out at 95°, and was brought into an office that was held at a constant 20°. After 10 minutes, the temperature of the coffee was 90°. Assume that Newton's Law of Cooling applies. How much additional time must elapse until the coffee reaches 80°?