Calculus 12

Integrated Resource Package (2000)

Extracted from

Mathematics 10 to 12 Integrated Resource Package (2000)

July 2012

Please disregard all references to

Applications of Mathematics 10 to 12

Essentials of Mathematics 10-12 and

Principles of Mathematics 10-12

BRITISH COLUMBIA

This Integrated Resource Package (IRP) provides some of the basic information that teachers will require to implement Applications of Mathematics 10 to 12, Essentials of Mathematics 10 to 12, Principles of Mathematics 10 to 12, and Calculus 12. The information contained in this IRP is also available through the Internet. Contact the Ministry of Education's home page: http://www.bced.gov.bc.ca/

THE INTRODUCTION

The Introduction provides general information about the Grade 10 to 12 Mathematics curriculum, including special features and requirements. It also provides a rationale for the subject—why mathematics is taught in BC schools—and an explanation of the curriculum organizers.

THE GRADE 10 TO 12 MATHEMATICS CURRICULUM

The provincially prescribed curriculum for Grade 10 to 12 Mathematics is structured in terms of *curriculum organizers*. The main body of this IRP consists of four columns of information for each organizer. These columns describe:

- provincially prescribed learning outcome statements for Grade 10 to 12 Mathematics
- suggested instructional strategies for achieving the outcomes
- suggested assessment strategies for determining how well students are achieving the outcomes provincially
- recommended learning resources

Prescribed Learning Outcomes

Learning outcome statements are content standards for the provincial education system. Learning outcomes set out the knowledge, enduring ideas, issues, concepts, skills, and attitudes for each subject. They are statements of what students are expected to know and be able to do in each grade. Learning outcomes are clearly stated and expressed in measurable terms. All learning outcomes complete this stem: "It is expected that students will. . . ."

Outcome statements have been written to enable teachers to use their experience and professional judgment when planning and evaluating. The outcomes are benchmarks that will permit the use of criterion referenced performance standards. It is expected that actual student performance will vary. Evaluation, reporting, and student placement with respect to these outcomes depends on the professional judgment of teachers, guided by provincial policy.

Suggested Instructional Strategies

Instruction involves the use of techniques, activities, and methods that can be employed to meet diverse student needs and to deliver the prescribed curriculum. Teachers are free to adapt the suggested instructional strategies or substitute others that will enable their students to achieve the prescribed outcomes. These strategies have been developed by specialist and generalist teachers to assist their colleagues; they are suggestions only.

Suggested Assessment Strategies

The assessment strategies suggest a variety of ways to gather information about student performance. Some assessment strategies relate to specific activities; others are general. As with the instructional strategies, these strategies have been developed by specialist and generalist teachers to assist their colleagues; they are suggestions only.

Provincially Recommended Learning Resources

The Ministry of Education promotes the establishment of a resource-rich learning environment through the evaluation of educationally appropriate materials intended for use by teachers and students. The media formats include, but are not limited to, materials in print, video, and software, as well as combinations of these formats. Learning resources for Applications of Math 10 to 12 and Principles of Math 10 to 12 were reviewed and recommended as part of the Western Canadian Protocol (WCP) Mathematics Learning Resource Evaluation. The WCP recommended learning resources have been approved by the Minister and have been incorporated into the Grade Collection for each course.

The Grade Collection package contains a grade collection chart for Applications of Mathematics 10 to 12 and Principles of Mathematics 10 to 12. These charts list both the comprehensive and additional resources for each curriculum organizer for the course. Each chart is followed by an annotated bibliography. Please confirm with suppliers for complete and up-to-date information.

Provincially recommended learning resources for Calculus 12 are materials that have been reviewed and evaluated by British Columbia teachers in collaboration with the Ministry of Education.

Provincially recommended learning resources for Essentials of Mathematics 10 to 12 are being developed and will be identified at a later date. As an interim measure the ministry has provided schools with photocopy masters of learning resources developed for a similar curriculum.

It is expected that teachers will select resources from those that meet the provincial criteria and that suit their particular pedagogical needs and audiences. Teachers who wish to use non-provincially recommended resources to meet specific local needs must have these resources evaluated through a local district approval process.

THE APPENDICES

A series of appendices provides additional information about the curriculum, and further support for the teacher.

- ***Appendix A*** contains a listing of the prescribed learning outcomes for the curriculum.
- ***Appendix B*** contains a comprehensive listing of the provincially recommended learning resources for this curriculum. As new resources are evaluated, this appendix will be updated.
- ***Appendix C*** outlines the cross-curricular reviews used to ensure that concerns such as equity, access, and the inclusion of specific topics are addressed by all components of the IRP.
- ***Appendix D*** contains assistance for teachers related to provincial evaluation and reporting policy. Prescribed learning outcomes have been used as the source for examples of criterion-referenced evaluations.
- ***Appendix E*** acknowledges the many people and organizations that have been involved in the development of this IRP.
- ***Appendix F*** contains an illustrated glossary of terms used in this IRP.
- ***Appendix G*** contains illustrative examples which show the competencies an average student is expected to demonstrate for each of the prescribed learning outcomes.
- ***Appendix H*** contains more detailed information related to specific teaching strategies referred to in the IRP. This information is provided in cases where explanations of strategies and activities are too extensive to fit in the Suggested Instructional Strategies column, or where the same activity is referred to several times.

Grade

Curriculum Organizer

PRINCIPLES OF MATHEMATICS 10 • *Patterns and Relations (Patterns)*

Prescribed Learning Outcomes

The Prescribed Learning Outcomes column of this IRP lists the specific learning outcomes for each curriculum organizer.

PRESCRIBED LEARNING OUTCOMES

It is expected that students will generate and analyse number patterns.

It is expected that students will:

- generate number patterns exhibiting arithmetic growth
- use expressions to represent general terms and sums for arithmetic growth, and apply these expressions to solve problems
- relate arithmetic sequences to linear functions defined over the natural numbers
- generate number patterns exhibiting geometric growth

SUGGESTED INSTRUCTIONAL STRATEGIES

Many natural phenomena and man-made processes grow and shrink according to easily defined mathematical rules. An understanding of arithmetic and geometric growth helps students to better understand how the world around them changes.

- Give students a collection of arithmetic or geometric sequences with blanks either between the numbers or at the end of the sequence for students to fill in. Have them state the common difference or ratio and give a word-based rule describing how one term is calculated from the previous term.
- Have students find three examples of arithmetic or geometric sequences found in nature or used by people in their day-to-day lives.
- Ask students to match a column of linear equations to a column of arithmetic sequences.
- Have students describe or model natural phenomena that incorporate geometric sequences in their structure (e.g., flowers, spiral shells, paper folding).
- Give students numerical and non-numerical patterns and ask them to define the rule(s) for each pattern. Have them use the rules to make predictions and classify the patterns.
- Have students use concepts of arithmetic and geometric growth to solve problems such as:
  - If $1000 is deposited each year into an account that earns 12% compounded annually, how much money will have accumulated after 25 years?

  Then have them:
  - generate a model of this situation
  - identify it as geometric or arithmetic
  - discuss the answer (Is it surprising? Reasonable?)

Suggested Instructional Strategies

The Suggested Instructional Strategies column of this IRP suggests a variety of instructional approaches that include group work, problem solving, and the use of technology. Teachers should consider these as examples that they might modify to suit the developmental levels of their students.

Grade

Curriculum Organizer

PRINCIPLES OF MATHEMATICS 10 • *Patterns and Relations (Patterns)*

Suggested Assessment Strategies

The Suggested Assessment Strategies offer a wide range of different assessment approaches useful in evaluating the prescribed learning outcomes. Teachers should consider these as examples they might modify to suit their own needs and the instructional goals.

SUGGESTED ASSESSMENT STRATEGIES

Assessment strategies should demonstrate algebraic skills as well as holistic pattern recognition. Students should also be able to articulate methods to aid in pattern discovery.

Observe

- How quickly do students see the patterns in partially completed sequences:
  - with no hints (2, 5, 8…)
  - with hints (2, _, _, 16, _… - geometric)
- Can students develop a mathematical equation from a sequence?

Question

- Can students articulate what to look for in determining whether a collection of numbers is an arithmetic sequence, geometric sequence, or neither?

Collect

- Have students collect five arithmetic, five geometric, and five other sequences. Provide them with equations or descriptions of how to generate each sequence.

Peer/Self Assessment

- Have each student develop a connect-the-dots puzzle using a combination of sequences and linear equations. Ask them to have another student do the puzzle and check it for accuracy.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Exploring Advanced Algebra with the TI-83
- An Introduction to the TI-82 Graphing Calculator
- Modeling Motion: High School Math Activities with the CBR
- Pure Mathematics 10 (Distance Learning Package)
- A Visual Approach to Algebra

Multi-media

- The Learning Equation: Mathematics 10 Lessons 22 - 24
- Mathematics 10, Western Canadian Edition Ch. 1 (Sections 1.1, 1.2, 1.4) Ch. 5 (Section 5.5)
- MATHPOWER 10, Western Edition Ch. 2 (Sections 2.3, 2.5, 2.7)

Software

- Secondary Math Lab Toolkit
- Understanding Math Series

Games/Manipulatives

- Radical Math: Math Games Using Cards and Dice (Volume VII)

CD-ROM

- Geometry Blaster

Recommended Learning Resources

The Recommended Learning Resources component of this IRP is a compilation of provincially recommended resources that support the prescribed learning outcomes. A complete list including a short description of the resource, its media type, and distributor is included in Appendix B of this IRP.

This Integrated Resource Package (IRP) sets out the provincially prescribed curriculum for the Grade 10 to 12 Mathematics curriculum. The development of this IRP has been guided by the principles of learning:

- Learning requires the active participation of the student
- People learn in a variety of ways and at different rates
- Learning is both an individual and a group process

THE DEVELOPMENT OF THIS INTEGRATED RESOURCE PACKAGE

A variety of resources were used in the development of this IRP:

- The prescribed learning outcomes, suggested instructional strategies, suggested assessment strategies, and Illustrative Examples for Applications of Mathematics 10 to 12 and Principles of Mathematics 10 to 12 were developed with reference to *The Common Curriculum Framework for K to 12 Mathematics* (Western Canadian Protocol for Collaboration in Basic Education, 1996)

- The prescribed learning outcomes, suggested instructional strategies, suggested assessment strategies, and Illustrative Examples for Essentials of Mathematics 10 to 12 were developed with reference to *Consumer Mathematics (Senior 2, 3, & 4): A Foundation for Implementation and Student Handbooks* (Manitoba Education & Training, 1999)

- Other written resources used include
  - *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989)
  - *Guidelines for Student Reporting*
  - provincial reference sets *Evaluating Problem Solving Across Curriculum* and *Evaluating Mathematical Development Across Curriculum*
  - *Report of the BCAMT Shape/Space (Geometry) Sub-Committee* (BC Association of Mathematics Teachers, 1999)
  - *Assessment Handbook Series*
  - *Summary of Responses* to the *Draft Grade 10 to 12 Mathematics IRP*
  - *Mathematics Proficiencies for Post-Secondary Mathematics/Statistics Courses: Project Report* (Neufeld, 1999)

This IRP represents the ongoing effort of the province to provide education programs that put importance on high standards in education while providing equity and access for all learners. In addition to this print version, the Grade 10 to 12 Mathematics IRP will also be available in electronic format.

RATIONALE

Mathematics is increasingly important in our technological society. Students today require the ability to reason and communicate, to solve problems, and to understand and use mathematics. Development of these skills helps students become numerate.

> Numeracy can be defined as the combination of mathematical knowledge, problem solving and communication skills required by all persons to function successfully within our technological world. Numeracy is more than knowing about numbers and number operations. (British Columbia Association of Mathematics Teachers, 1998)

Becoming numerate involves developing the ability to explore, conjecture, reason logically, and use a variety of mathematical methods to solve problems. It also involves the development of self-confidence and the ability to use quantitative and spatial information in problem solving and decision making. As students develop their numeracy skills and concepts, they generally grow more confident and motivated in their mathematical explorations. This growth occurs as they learn to enjoy and value mathematics, to think analytically, and to understand and appreciate the role of mathematics in everyday life.

The provincial mathematics curriculum emphasizes the development of numeracy skills and concepts and their practical application in higher education and the workplace. The curriculum places emphasis on probability and statistics, reasoning and communication, measurement, and problem solving. To ensure that students are prepared for the demands of both further education and the workplace, the graduate years of the mathematics curriculum (Grades 11 and 12) help students develop a more sophisticated sense of numeracy. At the same time, the curriculum investigates the creative and aesthetic aspects of mathematics by exploring the connections between mathematics, art, and design.

Developing Positive Attitudes

Research, including provincial assessments, consistently indicates that there is a positive correlation between student attitudes and performance. Mathematics activities should be interesting and engaging, so that students will be more likely to take risks to develop their mathematical thinking. Classroom practice and teaching strategies should promote positive attitudes toward mathematics for all students.

Becoming Mathematical Problem Solvers

Problem solving is the cornerstone of mathematics instruction. Students must learn the skills of effective problem solving, which include the ability to:

- read and analyse a problem
- identify the significant elements of a problem
- select an appropriate strategy to solve a problem
- work alone or in groups
- verify and judge the reasonableness of an answer
- communicate solutions

Acquiring these skills can help students become reasoning individuals able to contribute to society.

As students move through the grades, the curriculum presents them with increasingly diverse and complex mathematical problems to solve. To encourage students' abilities to communicate, explore, create, adjust to changes, and actively acquire new knowledge throughout their lives, mathematical problem solving should evolve naturally out of their experiences and be an integral part of all mathematical activity. Effective problem solving consists of more than being able to solve many different types of problems. Students need to be able to solve mathematical problems that arise in any subject area and to draw upon skills developed in more than one area of mathematics. Becoming a mathematical problem solver requires a willingness to take risks and persevere when faced with problems that do not have an immediately apparent solution.

Communicating Mathematically

Mathematics is a language—a way of communicating ideas. Communication plays an important role in helping students build links between their informal, intuitive notions and the abstract language and symbolism of mathematics. Communication also plays a key role in helping students make important connections among physical, pictorial, graphic, symbolic, verbal, descriptive, and mental representations of mathematical ideas. All activities that help students explore, explain, investigate, describe, and justify their decisions promote the development of communication skills. The Kindergarten to Grade 12 mathematics curriculum emphasizes discussing, writing, and representing mathematical thinking in various ways.

Connecting and Applying Mathematical Ideas

Learning activities should help students understand that mathematics is a changing and evolving domain to which many cultural groups have contributed. Students become aware of the usefulness of mathematics when mathematical ideas are connected to everyday experiences. Learning activities should therefore help students relate mathematical concepts to real-world situations and allow them to see how one mathematical idea can help them understand others. This approach emphasizes that mathematics helps students solve problems, describe and model real-world phenomena, and communicate complex thoughts and information with conciseness and precision.

Reasoning Mathematically

Mathematics instruction should help students develop confidence in their abilities to reason and to justify their thinking. Students should understand that mathematics is not simply memorizing rules. Mathematics should make sense, be logical, and also be enjoyable. The ability to reason logically usually develops on a continuum from concrete to formal to abstract. Students use inductive reasoning when they make conjectures by generalizing from a pattern of observations; they use deductive reasoning when they test those conjectures. To develop mathematical reasoning skills, students require the freedom to explore, conjecture, validate, and convince others. It is important that their ability to reason is valued as much as their ability to find correct answers.

Using Technology

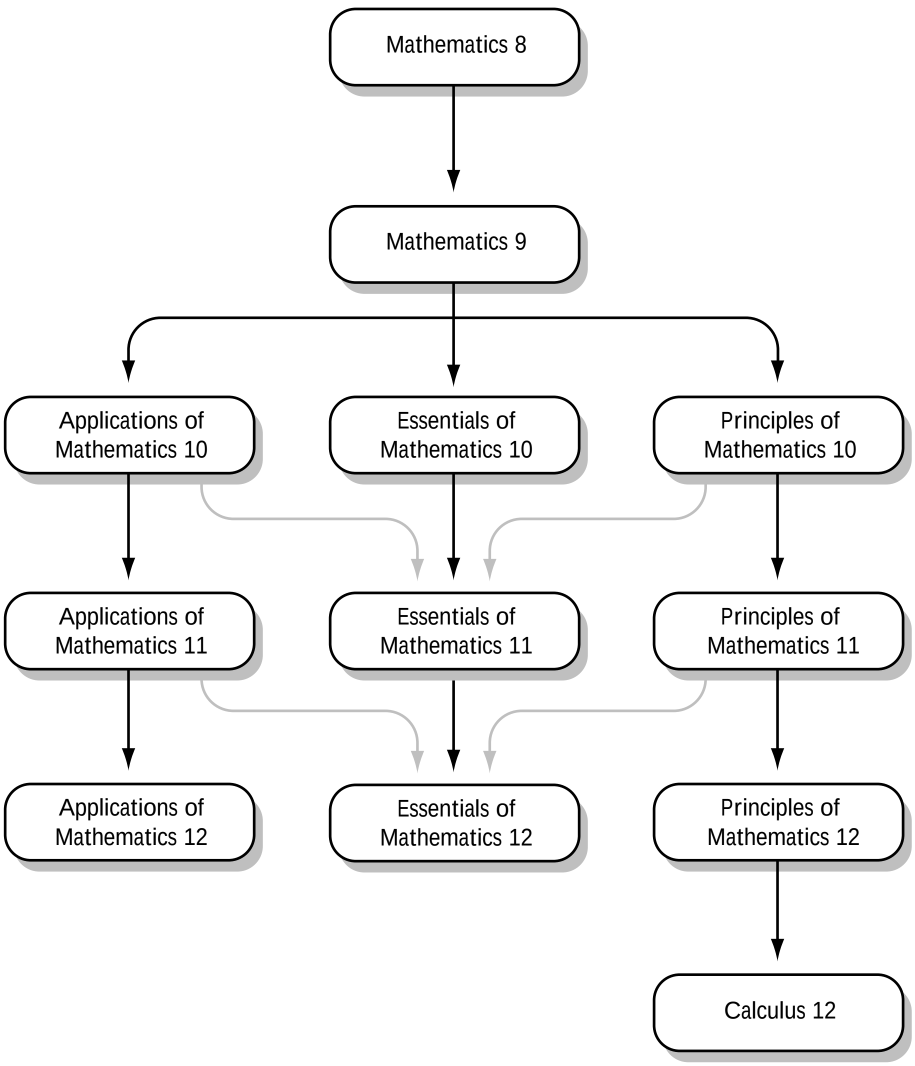
The Grade 10 to 12 Mathematics curriculum requires students to be proficient in using technology as a problem-solving tool. New technology has changed the level of sophistication of mathematical problems encountered today as well as the methods that mathematicians use to investigate them. Graphing tools such as computers and calculators are powerful aids to problem solving. The power to compute rapidly and to graph mathematical relationships instantly can help students explore many mathematical concepts and relationships in greater depth. When students have opportunities to use technology, their growing curiosity can lead to richer mathematical invention.

It is important to recognize that calculators and computers are tools that simplify, but do not accomplish, the work at hand. The availability of calculators does not eliminate the need for students to learn basic facts and algorithms. Students must have access to and be able to select and use the most appropriate tool or method for a calculation. In each of the mathematics courses, technology is used extensively to assist in the investigation and exploration of mathematical concepts.

Estimation and Mental Math

Mathematics involves more than exactness. Estimation strategies help students deal with everyday quantitative situations. Estimation skills also help them gain confidence and enable them to determine if something is mathematically reasonable. Even though they may have access to calculators from Kindergarten to Grade 12, students need to use reasoning, judgment, and decision-making strategies when estimating. Instruction should therefore emphasize the role that these strategies play.

B.C. SECONDARY MATHEMATICS COURSE STRUCTURE



Note: To simplify this diagram, not all possible student transitions between the Applications of Mathematics Pathway, the Essentials of Mathematics Pathway, and the Principles of Mathematics Pathway have been shown.

The Three Pathways

The mathematics curriculum for Grades 10 to 12 offers students a choice of routes through the different mathematics courses offered. Although each student's exact route will depend on a variety of factors, there are three main pathways:

- Applications of Mathematics
- Essentials of Mathematics
- Principles of Mathematics and Calculus 12

The Applications of Mathematics Pathway

The Applications of Mathematics pathway provides a practical, contextual focus that encourages students to develop their mathematical knowledge, skills, and attitudes in the context of their lives and possible careers. The instructional approaches used to develop the required mathematical concepts emphasize concrete activities and modeling, with less emphasis on symbol manipulation. When needed, students should have access to technology that extends their basic skills and knowledge and allows them to repeatedly investigate and model mathematical concepts and issues.

Students from the Applications of Mathematics pathway will be well prepared for many post-secondary programs that do not require calculus as part of the program of studies. The breadth of the Applications of Mathematics curriculum is intended to prepare students for entrance into many certificate programs, diploma programs, continuing education programs, trades programs, technical programs, and some degree programs.

The Essentials of Mathematics Pathway

In order to meet the challenges of society, high school graduates must be numerate. Students following this pathway will have opportunities to improve their numeracy skills and concepts. Developing a sense of numeracy will help them to understand how mathematical concepts permeate daily life, business, industry, and government. Students need to be able to use mathematics not just in their work lives, but in their personal lives as citizens and consumers. It is intended that students will learn to value mathematics and become confident in their mathematical abilities.

The Principles of Mathematics Pathway

Students following the Principles of Mathematics pathway will spend more time developing their understanding of symbol manipulation and of generalizations of more sophisticated mathematical concepts. Although there is an increased focus in this pathway on the applications of mathematics, one of the primary purposes of Principles of Mathematics will be to develop the formalism students will need to continue on with the study of calculus.

Both Applications of Mathematics 12 and Principles of Mathematics 12 have a provincial exam component. Students who successfully complete Applications of Mathematics 11, Essentials of Mathematics 11, or Principles of Mathematics 11 will meet British Columbia's graduation requirements.

Calculus 12

Calculus 12 is intended for students who have completed (or are concurrently taking) Principles of Mathematics 12 or who have completed an equivalent college preparatory course that includes algebra, geometry, and trigonometry.

Students taking Calculus 12 should be prepared to write the UBC - SFU - UVic - UNBC Challenge Examination if they choose to do so. For more information concerning the Challenge Examination contact the Mathematics Department at one of these universities.

Some schools may choose to develop articulation agreements with their local colleges. Students under these agreements may receive credit for first-term calculus (depending upon the particular agreement).

Calculus 12 Prerequisites

In *Mathematics Proficiencies for Post-Secondary Mathematics/Statistics Courses: Project Report* (Neufeld, 1999), the following concepts and skills (listed in descending order of importance) were identified as "essential" to "marginally important" for students to possess in order to attempt a course in calculus:

- the function concept
- polynomial expressions
- exponential expressions
- straight line and linear functions
- solving equations and inequalities
- circular trigonometric functions
- rational expressions
- triangle trigonometry
- the quadratic function
- logarithmic function
- radical expressions
- the geometry of lines and points
- polynomial functions
- quadratic relations
- sequences and series
- the geometry of circles

Teachers are urged to assess their students' mathematics proficiency as it relates to these topics so that any deficiencies can be addressed before new calculus concepts are taught.

ORGANIZATION OF THE CURRICULUM

The prescribed learning outcomes for the courses described in this Integrated Resource Package are grouped under a number of curriculum organizers. These curriculum organizers reflect the main areas of mathematics that students are expected to address. They form the framework of the curriculum and act as connecting threads across all grade levels for each pathway. The organizers are not equivalent in terms of number of outcomes or the time that students will require in order to achieve these outcomes. Suggestions for estimated instructional times have been included in this IRP. Teachers are expected to adjust these estimated instructional times to meet their students' diverse needs.

Within each course, the prescribed learning outcomes under many of the curriculum organizers are grouped under one, two, or three suborganizers. Each set of prescribed learning outcomes is introduced by a broad statement of the associated general learning outcomes for mathematics. (Material related to the general outcomes or suborganizers is not addressed in every course.)

The ordering of organizers and outcomes in the Grade 10 to 12 mathematics curriculum does not imply an order of instruction. The order in which various outcomes and topics are addressed is left to the professional judgment of teachers.

SUGGESTED INSTRUCTIONAL STRATEGIES

Instructional strategies have been included for each curriculum organizer (or suborganizer) and grade level. These strategies are suggestions only, designed to provide guidance for generalist and specialist teachers planning instruction to meet the prescribed learning outcomes. Some links to other subjects are indicated.

The strategies may be teacher directed, student directed, or both. There is not necessarily a one-to-one relationship between learning outcomes and instructional strategies, nor is this organization intended to prescribe a linear approach to course delivery; it is expected that teachers will adapt, modify, combine, and organize instructional strategies to meet the needs of students and respond to local requirements.

Context Statements

Each set of instructional strategies starts with a context statement followed by several examples of learning activities. The context statement links the prescribed learning outcomes with instruction. It also states why these outcomes are important for the student's mathematical development.

Instructional Activities

The mathematics curriculum is designed to emphasize the skills needed in the workplace, including those involving the use of probability and statistics, reasoning, communicating, measuring, and problem solving.

Additional emphasis is given to strategies and activities that:

- *foster the development of positive attitudes*

Students should be exposed to experiences that encourage them to enjoy and value mathematics, develop mathematical habits of mind, and understand and appreciate the role of mathematics in human affairs. They should be encouraged to explore, take risks, exhibit curiosity, and make and correct errors, so that they gain confidence in their abilities to solve complex problems. The assessment of attitudes is indirect, and based on inferences drawn from students' behaviour. We can see what students do and hear what they say, and from these observations make inferences and draw conclusions about their attitudes.

- *apply mathematics*

For students to view mathematics as relevant and useful, they must see how it can be applied to a wide variety of real-world applications. Mathematics helps students understand and interpret their world and solve problems that occur in their daily lives.

- *use manipulatives*

Using manipulatives is a good way to actively involve students in mathematics. Manipulatives encourage students to explore, develop, estimate, test, and apply mathematical ideas in relation to the physical world. Manipulatives range from commercially developed materials to simple collections of materials such as boxes, cans, or cards. They can be used to introduce new concepts or to provide a visual model of a mathematical concept.

- *use technology*

The use of technology in our society is increasing. Technological skills are becoming mandatory in the workplace. Instruction and assessment strategies that use a range of technologies such as calculators, computers, CD-ROMs, and videos will help students relate mathematics to their personal lives and prepare them for the future. The use of technology in developing mathematical concepts and as an aid in solving complex problems is encouraged to a greater extent as the student moves from grade to grade.

- *require problem solving*

For students to develop decision-making and problem-solving skills, they need learning experiences that challenge them to recognize problems and actively try to solve them, to develop and use various strategies, and to learn to represent solutions in ways appropriate to their purposes. Problems that occur within the students' environment can be used as the vehicle or context for students to achieve the learning outcomes in any of the curriculum organizers.

Instructional Focus

The Grade 10 to 12 mathematics courses are arranged into a number of organizers, including the Problem Solving organizer. Decreasing emphasis on rote calculation, drill and practice, and the size of numbers used in paper and pencil calculations allows more time for concept development.

In addition to problem solving, other critical thinking processes—reasoning and making connections—are vital to increasing students' mathematical power and must be integrated throughout the program. A minimum of half the available time within all organizers should be dedicated to activities related to these processes.

Instruction should provide a balance between estimation and mental mathematics, paper and pencil exercises, and the appropriate use of technology, including calculators and computers. (It is assumed that all students have regular access to appropriate technology such as graphing calculators, or computers with graphing software and standard spreadsheet programs.) Concepts should be introduced using manipulatives, and gradually developed from the concrete to the pictorial to the symbolic.

INTEGRATION OF CROSS-CURRICULAR INTERESTS

Integration of Cross-Curricular Interests

Throughout the curriculum development and revision process, the advice of experts has been invited to ensure that relevance, equity, and accessibility issues are addressed in all Integrated Resource Packages.

The recommendations of these cross-curricular reviews have been integrated into the prescribed learning outcomes, suggested instructional strategies, and assessment strategies components of all curricula with respect to the following:

- Applied Focus
- Career Development
- Multiculturalism and Anti-Racism
- English as a Second Language (ESL)
- Special Needs
- Aboriginal Studies
- Gender Equity
- Information Technology
- Media Education
- Science-Technology-Society
- Environment and Sustainability

See Appendix C: Cross-Curricular Interests for more information.

Gender Issues in Mathematics

The education system is committed to helping both male and female students succeed equally well. In British Columbia, significant progress has been made in improving the participation and success rate of female students in secondary math courses. They now take about the same number of secondary math courses as males. There continues, however, to be a relatively low rate of female participation in math-related careers and education. Positive attitudes toward the practice of mathematics, as well as skill in mathematics, are essential to the workplace and to everyone's ability to participate fully in society. Teaching, assessment materials, learning activities, and classroom environments should place value on the mathematical experiences and contributions of both men and women and people of diverse cultures.

Research regarding gender and mathematics has raised a number of important issues that teachers should consider when teaching mathematics. These include the diversity of learning styles, gender bias in learning resources, and unintentional gender bias in teaching. The following instructional strategies are suggested to help the teacher deliver a gender-sensitive mathematics curriculum.

- Feature both females and males who are mathematicians, or who make extensive use of mathematics in their careers, as guest speakers or subjects of study in the classroom.
- Design instruction to acknowledge differences in experiences and interests between young women and young men.
- Demonstrate the relevance of mathematics to a variety of careers and to everyday life in ways that are apt to appeal to particular students in the class or school. Successful links include biology, environmental issues, and current topics in mass media.
- Explore not only the practical applications of mathematics, but also the human elements, such as ways in which ideas have changed throughout history and the social and moral implications of mathematics.
- Explore ways of approaching mathematics that will appeal to a wide variety of students. Use co-operative rather than competitive instructional strategies. Focus on concept development, encouraging students to question until they can say "I've got it." Include a wide variety of applications that demonstrate the role of math in the social fabric of our world. Varying approaches appeal to a wider variety of students.
- Emphasize that mathematics is used by ordinary people with a variety of interests and responsibilities.
- Allowing for informal social interaction with successful "math using" members of the community will help change the negative stereotypes of mathematicians and their social style.
- Provide opportunities for visual and hands-on activities, which most students enjoy. Experiments, demonstrations, field trips, and exercises that provide opportunities to explore the relevance of mathematics are particularly important.

Adapting Instruction for Diverse Student Needs

When students with special needs are expected to achieve or surpass the learning outcomes set out in the mathematics curriculum, regular grading practices and reporting procedures are followed. However, when students are not expected to achieve the learning outcomes, adaptations and modifications must be noted in their Individual Education Plans (IEPs). Instructional and assessment methods should be adapted to meet the needs of all students.

The following strategies may help students with special needs succeed in mathematics:

- Adapt the Environment
  - Change the student's seat in the classroom.
  - Make use of co-operative grouping.
- Adapt Presentations
  - Provide students with advance organizers of the key mathematical concepts.
  - Demonstrate or model new concepts.
  - Adapt the pace of activities as required.
- Adapt Materials
  - Use techniques, such as colour coding the steps to solving a problem, to make the organization of activities more explicit.
  - Use manipulatives such as large-size dice, cards, and dominoes.
  - Use large-print charts such as a 100s chart or a times-table chart.
  - Provide students with a talking calculator or a calculator with a large keypad.
  - Use large print on activity sheets.
  - Use opaque overlays on text pages to reduce the quantity of print that is visible.
  - Highlight key points on activity sheets.
- Adapt Methods of Assistance
  - Have peers or volunteers assist students with special needs.
  - Have students with special needs help younger students learn mathematics.
  - Have teacher assistants work with individuals and small groups of students with special needs.
  - Work with consultants and support teachers to develop problem-solving activities and strategies for
  - mathematics instruction for students with special needs.
- Adapt Methods of Assessment
  - Allow students to demonstrate their understanding of mathematical concepts in a variety of ways, such as murals, displays, models, puzzles, game boards, mobiles, and tape recordings.
  - Modify assessment tools to match student needs. For example, oral tests, open-book tests, and tests with no time limit may allow students to better demonstrate their learning than a traditional timed paper-and-pencil test.
  - Set achievable goals.
  - Use computer programs that provide opportunities for students to practise mathematics as well as record and track their results.
- Provide Opportunities for Extension and Practice
  - Require the completion of only a small amount of work at a given time.
  - Simplify the way questions are worded to match the student's level of understanding.
  - Provide functional, everyday contexts (e.g., cooking) in which students can practise measurement skills.

SUGGESTED ASSESSMENT STRATEGIES

Teachers determine the best assessment methods for their students. The assessment strategies in this document describe a variety of ideas and methods for gathering evidence of student performance. The assessment strategies column for a particular organizer always includes specific examples of assessment strategies. Some strategies relate to particular activities, while others are general and could apply to any activity. These specific strategies may be introduced by a context statement that explains how students at this age can demonstrate their learning, what teachers can look for, and how this information can be used to adapt further instruction.

About Assessment in General

Assessment is the systematic process of gathering information about students' learning in order to describe what they know, are able to do, and are working toward. From the evidence and information collected in assessments, teachers describe each student's learning and performance. They use this information to provide students with ongoing feedback, plan further instructional and learning activities, set subsequent learning goals, and determine areas requiring diagnostic teaching and intervention. Teachers base their evaluation of a student's performance on the information collected through assessment.

Teachers determine the purpose, aspects, or attributes of learning on which to focus the assessment; when to collect the evidence; and the assessment methods, tools, or techniques most appropriate to use. Assessment focuses on the critical or significant aspects of the learning to be demonstrated by the student.

The assessment of student performance is based on a wide variety of methods and tools, ranging from portfolio assessment to pencil-and-paper tests. Appendix D includes a more detailed discussion of assessment and evaluation.

CURRICULUM

Calculus 12

ESTIMATED INSTRUCTIONAL TIME

Calculus 12 has been developed assuming that teachers have 100 instructional hours available to them. The following chart shows the estimated instructional time for each curriculum suborganizer, expressed as a percentage of total time available to teach the course.

Calculus 12

Organizer (Suborganizer)	% of Time
Problem Solving	**Integrated Throughout**
Overview and History of Calculus	**Integrated Throughout**
Functions, Graphs, and Limits (Functions and their Graphs)(Limits)	**10 - 15**
The Derivative (Concept and Interpretations)	**10 - 15**
The Derivative (Computing Derivatives)	**15 - 20**
Applications of Derivatives (Derivatives and the Graph of the Function)	**15 - 20**
Applications of Derivatives (Applied Problems)	**20 - 25**
Antidifferentiation (Recovering Functions from their Derivatives)	**5 - 10**
Antidifferentiation (Applications of Antidifferentiation)	**10 - 15**

When delivering the prescribed curriculum, teachers may freely adjust the instructional time to meet their students' diverse needs. These estimated instructional times have been recommended by the IRP writers to assist their colleagues; they are suggestions only.

PRESCRIBED LEARNING OUTCOMES

It is expected that students will use a variety of methods to solve real-life, practical, technical, and theoretical problems.

It is expected that students will:

- solve problems that involve a specific content area (e.g., geometry, algebra, trigonometry statistics, probability)
- solve problems that involve more than one content area
- solve problems that involve mathematics within other disciplines
- analyse problems and identify the significant elements
- develop specific skills in selecting and using an appropriate problem-solving strategy or combination of strategies chosen from, but not restricted to, the following:
  - guess and check
  - look for a pattern
  - make a systematic list
  - make and use a drawing or model
  - eliminate possibilities
  - work backward
  - simplify the original problem
  - develop alternative original approaches
  - analyse keywords
- demonstrate the ability to work individually and co-operatively to solve problems
- determine that their solutions are correct and reasonable
- clearly communicate a solution to a problem and justify the process used to solve it
- use appropriate technology to assist in problem solving

SUGGESTED INSTRUCTIONAL STRATEGIES

Problem solving is a key aspect of any mathematics course. Working on problems can give students a sense of the excitement involved in creative and logical thinking. It can also help students develop transferable real-life skills and attitudes. Problems may come from various fields, including algebra, geometry, and statistics. Multi-strand and interdisciplinary problems should be included throughout Calculus 12.

- Introduce new types of problems directly to students (without demonstration) and play the role of facilitator as they attempt to solve such problems.
- Recognize when students use a variety of approaches (e.g., algebraic and geometric solutions); avoid becoming prescriptive about approaches to problem solving.
- Reiterate that problems might not be solved in one sitting and that "playing around" with the problem-revisiting it and trying again-is sometimes needed.
- Frequently engage small groups of students (three to five) in co-operative problem solving when introducing new types of problems.
- Have students or groups discuss their thought processes as they attempt a problem. Point out the strategies inherent in their thinking (e.g., guess and check, look for a pattern, make and use a drawing or model).
- Ask leading questions such as:
  - What are you being asked to find out?
  - What do you already know?
  - Do you need additional information?
  - Have you ever seen similar problems?
  - What else can you try?
- Once students have arrived at solutions to particular problems, encourage them to generalize or extend the problem situation.

SUGGESTED ASSESSMENT STRATEGIES

Students analyse problems and solve them using a variety of approaches. Assessment of problem-solving skills is made over time, based on observations of many situations.

Observe

- Have students present solutions to the class individually, in pairs, or in small groups. Note the extent to which they clarify their problems and how succinctly they describe the processes used.

Question

- To check the approaches students use when solving problems, ask questions that prompt them to:
  - paraphrase or describe the problem in their
  - own words
  - explain the processes used to derive an answer
  - describe alternative methods to solve a problem
  - relate the strategies used in new situations
  - link mathematics to other subjects and to the world of work

Collect

- On selected problems, have students annotate their work to describe the processes they used. Alternatively, have them provide brief descriptions of what worked and what did not work as they solved particular problems.

Self-Assessment

- Ask students to keep journals to describe the processes they used in dealing with problems. Have them include descriptions of strategies that worked and those that did not.
- Develop with students a set of criteria to self-assess problem-solving skills. The reference set Evaluating Problem Solving Across Curriculum may be helpful in identifying such criteria.

RECOMMENDED LEARNING RESOURCES

Please see the introduction to Appendix B for a list of suggested utility software that supports this course.

The Western Canadian Protocol Learning Resource Evaluation Process has also identified numerous *teacher resources* and *professional references*. These are generally cross-grade planning resources that include ideas for a variety of activities and exercises.

These resources, while not part of the Grade Collections, have Provincially Recommended status.

Appendix B includes an annotated bibliography of these resources for ordering convenience.

Prescribed Learning Outcomes

It is expected that students will understand that calculus was developed to help model dynamic situations.

It is expected that students will:

- distinguish between static situations and dynamic situations
- identify the two classical problems that were solved by the discovery of calculus:
  - the tangent problem
  - the area problem
- describe the two main branches of calculus:
  - differential calculus
  - integral calculus
- understand the limit process and that calculus centers around this concept

Suggested Instructional Strategies

Students will quickly come to realize that calculus is very different from the mathematics they have previously studied. Of greatest importance is an understanding that calculus is concerned with change and motion. It is a mathematics of change that enables scientists, engineers, economists, and many others to model real-life, dynamic situations.

- The concepts of average and instantaneous velocity are a good place to start. These concepts could be related to the motion of a car. Ask students if the displacement of the car is given by $f(t)$, what is the average velocity of the car between times $t = t_1$ and $t = t_2$, and the instantaneous velocity at $t = t_1$? Point out that calculus is not needed to determine the average velocity $\frac{f(t_2) - f(t_1)}{t_2 - t_1}$, but is needed to determine the velocity at t_1 (the speedometer reading).
- As a way of solving the tangent problem (i.e., the need for two points and the fact that only one point is available), create a secant line that can be moved toward the tangent line. Ask students if they can move the secant line towards the tangent line using algebraic techniques.
- Ask students to determine the area under a curve by estimating the answer using rectangles. Introduce the concept of greater accuracy when the rectangles are of smaller and smaller widths (the idea of limit).
- Present the following statement for discussion:

  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$

  Use geometry to illustrate the relationship between the left-hand and right-hand sides.

SUGGESTED ASSESSMENT STRATEGIES

To demonstrate their achievement of the outcomes for this organizer, students need opportunities to engage in open-ended activities that allow for a range of responses and representations.

Although formative assessment of students' achievement of the outcomes related to this organizer should be ongoing, summative assessment can only be carried out effectively toward the end of the course, when students have sufficient understanding of the details of calculus to appreciate the "big picture."

Self-Assessment

- Work with the students to develop criteria and rating systems they can use to assess their own descriptions of the two main branches of calculus. They can represent their progress with symbols to indicate when they have addressed a particular criterion in their work. Appropriate criteria might include the extent to which they:
  - make connections to other branches of mathematics
  - demonstrate understanding of the classical problems
  - consider the different attempts to solve the classical problem(s)
  - explain the significance of limits with reference to specific examples
  - accurately and comprehensively describe the relationship between integral and differential calculus (methodology and purpose)

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  pp. 83-85, 95, 190, 262, 272-273,375
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 57 - 61, 63
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 2 (Section 2.2)
  pp. 3-4, 6-9, 85, 351

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 2, 112, 114, 375
- Calculus of a Single Variable, Sixth Edition
  Ch. 1 (Section 1.1)
  Ch. 4 (Section 4.2, 4.3
  pp. 41-45, 91, 265

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand the historical background and problems that lead to the development of calculus.

It is expected that students will:

- describe the contributions made by various mathematicians and philosophers to the development of calculus, including:
  - Archimedes
  - Fermat
  - Descartes
  - Barrow
  - Newton
  - Leibniz
  - Jakob and Johann Bernoulli
  - Euler
  - L'Hospital

SUGGESTED INSTRUCTIONAL STRATEGIES

Students gain a better understanding and appreciation for this field of mathematics by studying the lives of principal mathematicians credited for the invention of calculus, including the period in which they lived and the significant mathematical problems they were attempting to solve.

- Conduct a brief initial overview of the historical development of calculus, then deal with specific historical developments when addressing related topics (e.g., the contributions of Fermat and Descares to solving the tangent line problem can be covered when dealing with functions, graphs, and limits). For additional ideas, see the suggested instructional strategies for other organizers.
- Encourage students to access the Internet for information on the history of mathematics.
- Ask students to investigate various mathematicians and the associated periods of calculus development and to present their findings to the class.
- Point out the connection between integral and differential calculus when students are working on the "area under the curve" problem.
- Conduct a class discussion about the contributions of Leibniz and Newton when the derivative is being introduced (e.g., the notation $\frac{dy}{dx}$ for the derivative and $\int y\,dx$ for the integral are due to Leibniz; the $f'(x)$ notation is due to Lagrange).
- Have students research the historical context of the applications of antidifferentiation. For example:
  - The area above the x axis, under $y = x^2$, from $x = 0$ to $x = b$ is $b^3/3$. Archimedes was able to show this without calculus, using a difficult argument.
  - The area problem for $y = \frac{1}{x}$ was still unsolved in the early 17th century. Using antiderivatives, the area under the curve $y = \frac{1}{t}$ from $t = 1$ to $t = x$ is $\ln x$.
  - The area under $y = \sin x$, from $x = 0$ to $x = \frac{\pi}{2}$, was calculated by Roberval, without calculus, using a difficult argument. It can be solved easily using antiderivatives.

SUGGESTED ASSESSMENT STRATEGIES

When students are aware of the outcomes they are responsible for and the criteria by which their work will be assessed, they can represent their comprehension of the historical development of calculus more creatively and effectively.

Observe

- Have students construct and present a calculus timeline, showing important discoveries and prominent mathematicians. Check the extent to which students' work is:
  - accurate
  - complete
  - clear

Collect

- Ask students to write a short article about one mathematician's contributions to calculus. The article should describe the mathematical context within which the person lived and worked and the contributions as seen in calculus today.

Research

- Use a research assignment to assess students' abilities to synthesize information from more than one source. Have each student select a topic of personal interest, develop a list of three to five key questions, and locate relevant information from at least three different sources. Ask students to summarize what they learn by responding to each of the questions in note form, including diagrams if needed. Look for evidence that they are able to:
  - combine the information, avoiding duplications or contradictions
  - make decisions about which points are most important

Peer Assessment

- To check on students' knowledge of historical figures, form small groups and ask each group to prepare a series of three to five questions about the contribution of a particular mathematician. Have groups exchange questions, then discuss, summarize, and present their answers. For each presentation, the group that designed the questions offers feedback on the extent to which the answers are thorough, logical, relevant, and supported by specific explanation of the mathematics involved.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Integrated Throughout
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  Integrated Throughout
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Integrated Throughout

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 10-11, 15, 19, 99, 225, 237, 278, 352, 378, 638, A3
- Calculus of a Single Variable, Sixth Edition
  Integrated Throughout

PRESCRIBED LEARNING OUTCOMES

It is expected that students will represent and analyse rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions, using technology as appropriate.

It is expected that students will:

- model and apply inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions to solve problems
- draw (using technology), sketch and analyse the graphs of rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions, for:
  - domain and range
  - intercepts
- recognize the relationship between a base a exponential function ($a > 0$) and the equivalent base e exponential function (convert $y = a^x$ to $y = e^{x(1na)}$)
- determine, using the appropriate method (analytic or graphing utility) the points where $f(x) = 0$

SUGGESTED INSTRUCTIONAL STRATEGIES

With the aid of technology, graphs of functions are often easy to produce. The emphasis is on the interplay between the geometric and analytic information and on the use of calculus both to predict and to explain the observed local and global behavior of a function.

- Prior to undertaking work on functions and their graphs, conduct review activities to ensure that students can:
  - perform algebraic operations on functions and compute composite functions
  - use function and inverse function notation appropriately
  - determine the inverse of a function and whether it exists
  - describe the relationship between the domain and range of a function and its inverse
- Give students an exponential function and its inverse logarithmic function. Have them work in groups, using graphing calculators or computer software, to identify the relationship between the two. (e.g., compare and contrast the graphs of $\log_2 x$ and 2^x).
- Emphasize the domain and range restrictions as students work with exponential and natural logarithmic functions.
- Have students use technology (e.g., graphing calculators) to compare the graphs of functions such as:

  $y = e^x$ and $y = \ln x$

  $y = e^{\ln x}$ and $y = \ln e^x$

  $y = 3^x$ and $y = e^{x \ln 3}$
- Similarly apply this procedure to trigonometric and inverse trigonometric graphs including: sine and arcsine, cosine and arccosine, and tangent and arctangent. Emphasize that arcsine $x = \sin^{-1} x$ and does not mean the reciprocal of sin x.

SUGGESTED ASSESSMENT STRATEGIES

The exponential and natural logarithmic functions enable students to solve more complex problems in areas such as science, engineering, and finance. Students should be able to demonstrate their knowledge of the relationship between natural logarithms and exponential functions, both in theory and in application, in a variety of problem situations.

Observe

- Ask students to outline how they would teach a classmate to explain:
  - the inverse relationship between base *e* exponential and natural logarithmic functions
  - the restrictions associated with base *e* exponential and natural logarithmic, and trigonometric and inverse trigonometric functions

  Note the extent to which the outlines:
  - include general steps to follow
  - use mathematical terms correctly
  - provide clear examples
  - describe common errors and how they can be avoided

Collect

- Assign a series of problems that require students to apply their knowledge of the relationship between logarithms and exponential functions. Check their work for evidence that they:
  - clearly understood the requirements of the problem
  - used efficient strategies and procedures to solve the problem
  - recognized when a strategy or procedure was not appropriate
  - verified that their solutions were accurate and reasonable

Self-Assessment

- Discuss with students the criteria for assessing graphing skills. Show them how to develop a rating scale. Have them use the scale to assess their own graphing skills.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic pp. 12, 16, 23, 36-37, 45-51, 165
- Calculus of a Single Variable Early Transcendental Functions, Second Edition Ch. 1 (Sections 1.4, 1.5, 1.6) Ch. 4 (Section 4.6) pp. 5, 20, 26-27, 41, 51-52, 63, 158, 182
- Single Variable Calculus Early Transcendentals, Fourth Edition pp. 43-46, 61-63, 69-72, A31-A34

Multimedia

- Calculus: A New Horizon, Sixth Edition pp. 49, 57, 240, 242-243, 246, 254, 261, 308
- Calculus of a Single Variable, Sixth Edition Ch. 5 (Section 5.1, 5.4, 5.5, 5.8) pp. 20, 26, 27, 75, 134, 380

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand the concept of limit of a function and the notation used, and be able to evaluate the limit of a function.

It is expected that students will:

- demonstrate an understanding of the concept of limit and notation used in expressing the limit of a function $f(x)$ as x approaches a: $\lim_{x\to a} f(x)$
- evaluate the limit of a function
  - analytically
  - graphically
  - numerically
- distinguish between the limit of a function as x approaches a and the value of the function at $x = a$
- demonstrate an understanding of the concept of one-sided limits and evaluate one-sided limits
- determine limits that result in infinity (infinite limits)
- evaluate limits of functions as x approaches infinity (limits at infinity)
- determine vertical and horizontal asymptotes of a function, using limits
- determine whether a function is continuous at $x = a$

SUGGESTED INSTRUCTIONAL STRATEGIES

Students require a firm understanding of limits in order to fully appreciate the development of calculus.

- Although the concept of limit should be introduced at the beginning of the course, particular limits (e.g., $\lim_{x\to 0}\frac{\sin x}{x}$) need only be introduced as needed to deal with new derivatives.
- Using diagrams, introduce students in a general way to the two classic problems in calculus: the **tangent line problem** and the **area problem** and explain how the concept of limit is used in the analysis.
- Give students a simple function such as $f(x) = x^2$ and have them:
  - determine the slope of the tangent line to $f(x) = x^2$ at the point (2,4) by evaluating the slope of the secant lines through (2,4) and (2.5, f(2.5)), (2.1, f(2.1)), (2.05, f(2.05)), (2.01, f(2.01)), then draw conclusions about the value of the slope of the tangent line
  - determine the area under $y = x^2$ above the x axis from $x = 0$ to $x = 2$ by letting the number of rectangles be 4, 8, and 16, then inferring the exact area under the curve.
- Give students limit problems that call for analytic, graphical, and/or numerical evaluation, as in the following examples:
  - $\lim_{x\to 2}\frac{x^2-4}{x-2}$ (evaluate analytically)
  - $\lim_{x\to 0}\frac{\sin x}{x}$ (evaluate numerically, geometrically, and using technology)
  - $\lim_{x\to 0}\frac{3x^2+5}{4-x^2}$ (evaluate analytically and numerically)
  - $\lim_{x\to 0}\frac{1}{x^2}=\infty$ (draw conclusions numerically)
- When introducing students to one-sided limits that result in infinity, have them draw conclusions about the vertical asymptote to $y = f(x)$, as in the following example: from $\lim_{x\to 3^+}\frac{1}{x-3}$ and $\lim_{x\to 3^-}\frac{1}{x-3}$, conclude that $x = 3$ is the vertical asymptote for $f(x)=\frac{1}{x-3}$.
- Have students explore the limits to infinity of a function and draw conclusions about the horizontal asymptote, as in the following example: because $\lim_{x\to\infty}\frac{e^x}{e^x+1}=1$ and because $\lim_{x\to -\infty}\frac{e^x}{e^x+1}=0$, y=1 and y=0 are horizontal asymptotes for $f(x)=\frac{e^x}{e^x+1}$.

SUGGESTED ASSESSMENT STRATEGIES

Limits form the basis of how calculus can be used to solve previously unsolvable problems. Students should be able to demonstrate their knowledge of both the theory and application of limits in a variety of problem situations.

Observe

- While students are working on problems involving limits, look for evidence that they can:
  - distinguish between the limit of a function and the **value** of a function
  - Identify and evaluate one-sided limits
  - recognize and evaluate infinite limits and limits at infinity
  - determine any vertical or horizontal asymptote of a function
  - determine whether a function is continuous over a specified range or point
- To check students' abilities to reason mathematically, have them describe orally the characteristics of limits in relation to one-sided limits, infinite limits, limits at infinity, asymptotes, and continuity of a function. To what extent do they give explanations that are mathematically correct, logical, and clearly presented.

Collect

- Assign a series of problems that require students to apply their knowledge of limits. Check their work for evidence that they:
  - clearly understood the requirements of the problem
  - used the appropriate method to evaluate the limit in the problem
  - verified that their solutions were accurate and reasonable

Self-Assess/Peer Assess

- Have students explain concepts such as "limit" and "continuity" to each other in their own words.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic pp. 55-61, 65-77
- Calculus of a Single Variable Early Transcendental Functions, Second Edition pp. 63, 72, 75, 84, 96-97, 113, 227-228
- Single Variable Calculus Early Transcendentals, Fourth Edition Ch. 2 (Sections 2.2, 2.5, 2.6)

Multimedia

- Calculus: A New Horizon, Sixth Edition pp. 115, 118, 120, 122-124, 131, 148, 161
- Calculus of a Single Variable, Sixth Edition Ch. 1 (Section 1.2, 1.4, 1.5) Ch. 3 (Section 3.5)

PRESCRIBED LEARNING OUTCOMES

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

It is expected that students will:

- describe geometrically a secant line and a tangent line for the graph of a function at $x = a$
- define and evaluate the derivative at $x = a$ as: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$
- define and calculate the derivative of a function using:
  $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ or

  $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ or

  $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- use alternate notation interchangeably to express derivatives (i.e., $f'(x), \frac{dy}{dx}, y'$, etc.)
- compute derivatives using the definition of derivative
- distinguish between continuity and differentiability of a function at a point
- determine when a function is non-differentiable, and explain why
- determine the slope of a tangent line to a curve at a given point
- determine the equation of the tangent line to a curve at a given point
- for a displacement function $s = s(t)$, calculate the average velocity over a given time interval and the instantaneous velocity at a given time
- distinguish between average and instantaneous rate of change

SUGGESTED INSTRUCTIONAL STRATEGIES

The concept of the derivative can be used to calculate instantaneous rate of change. Students can best understand this concept if it is presented to them geometrically, analytically, and numerically. This allows them to make connections between the various forms of presentation.

- Present an introduction to the "tangent line" problem by discussing the contribution made by Fermat, Descartes, Newton, and Leibniz.
- Use a graphing utility to graph $f(x) = 2x^3 - 4x^2 + 3x - 5$.
  On the same screen, add the graphs of $y = x - 5$, $y = 2x - 5$, $y = 3x - 5$. Have students identify which of these lines appears to be tangent to the graph at the point (0, -5). Have them explain their answers.
- Sketch a simple parabola $y = x^2$ on the board. Attach a string at a point and show how it can be moved from a given secant line to a tangent line at a different point on the graph. Show how the slope of the secant line approaches that of the tangent line as $\Delta x \to 0$.
- Have students calculate the slope of a linear function at a point $x = a$ (e.g., $f(x) = 3x + 2$ at $x = 1$), using the formula $\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$. Have students use the same formula to find the slope of the tangent lines to the graph of a non-linear function such as $f(x) = x^2 - 4$ at $a = 1$, and then find the equation of the tangent lines.
- Ask students to use technology to:
  - reinforce the result found when using the definition to calculate the derivative of a function
  - verify that the tangent line as calculated, appears to be the tangent to the curve. As an extension, have students zoom in at the point of tangency and describe the relationship between the tangent line and the curve
- Introduce the differentiability of a function by generating problems that require students to create, using technology, graphs with sharp turns and vertical tangent lines.
- Have students work in groups to prepare a presentation on tangent lines and the use of mathematics by the ancient Greeks (circle, ellipse, parabola). Ask them to include an explanation of how the initial work of the Greeks was surpassed by the work of Fermat and Descartes and how previously difficult arguments were made into essentially routine calculations.

SUGGESTED ASSESSMENT STRATEGIES

The concept of the derivative facilitates the solving of complex problems in fields such as science, engineering, and finance. Students should be able to demonstrate knowledge of the derivative in relation to problems involving instantaneous rate of change.

Observe

- As students work, circulate through the classroom and note:
  - the extent to which they relate their algebraic work to their graphing work
  - whether they can recognize the declining nature of the x values
- Have students develop a table of values to calculate a "slope predictor" in the parabola $y = x^2$ at a point. In small groups they can justify the concept of a limit using technology. Higher magnifications will eliminate the difference in the slope values of secant and tangent lines at a point. Ask one student from each group to explain this phenomenon, using the group's 'unique' sample slope calculation. Note how succinctly they describe the processes used.
- When students work with technology (e.g., graphing calculator), check the extent to which they:
  - select appropriate viewing windows
  - enter the functions correctly
  - interpret the results correctly

Question

- Have students explain in their own words the concepts of average and instantaneous rates of change.

Research

- When students report on the contributions of various mathematicians to the tangent line problem, check the extent to which they:
  - clearly address key concepts (e.g., explain how Fermat found the length of the subtangent; how Descartes found the slope of the normal)
  - include careful graphic illustrations

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  pp. 83-85, 95-97, 105-107, 122-124, 152
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 60, 110, 112-117, 123, 125-129, 137, 142, 174, 662, 675
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 2 (Sections 2.7, 2.8, 2.9)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  Ch. 3
  pp. 170, 172, 175, 178-180, 186, 187, 353
- Calculus of a Single Variable, Sixth Edition
  Ch. 2 (Section 2.1)
  pp. 92-97, 108-109, 166

PRESCRIBED LEARNING OUTCOMES

It is expected that students will determine derivatives of functions using a variety of techniques.

It is expected that students will:

- compute and recall the derivatives of elementary functions including:
  - $\frac{d}{dx}(x^r) = rx^{r-1}$, r is real
  - $\frac{d}{dx}(e^x) = e^x$
  - $\frac{d}{dx}(\ln x) = \frac{1}{x}$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- use the following derivative formulas to compute derivatives for the corresponding types of functions:
  - constant times a function $\frac{d}{dx}cu = c\frac{du}{dx}$:
  - $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ (sum rule)
  - $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (product rule)
  - $\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (quotient rule)
  - $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$ (power rule)
- use the Chain Rule to compute the derivative of a composite function: $\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$ or $\frac{d}{dx}(F(g(x))) = g'(x)F'(g(x))$
- compute the derivative of an implicit function
- use the technique of logarithmic differentiation
- compute higher order derivatives

SUGGESTED INSTRUCTIONAL STRATEGIES

Knowledge of a variety of techniques for computing the derivatives of different types of functions allows students to solve an ever-increasing range of problems. To become efficient problem solvers students need to understand when and how to use derivative formulas, but will also find it useful to memorize some basic derivatives.

- Point out to students that the derivatives of a few functions (e.g., x^2, $\frac{1}{x}$, $\sqrt{x}$) will already have been obtained using the definition of a derivative.
- To help students connect the limit concept to the derivative of sin x, demonstrate the details of $\frac{d}{dx}(\sin x) = \cos x$, using the known fact that $\sin(x + h) = \cos x \sin h + \sin x \cos h$ and that $\lim_{h\to 0}\frac{\sin h}{h} = 1$. Alternatively, you can provide these known facts to students and have them work in groups to develop the solution.
- Have students work in groups to differentiate $f(x) = \frac{(1+x)^5(1-2x)^7}{x^3}$, using two different methods:
  1) the chain, product, and quotient rules
  2) logarithmic differentiation

  Ask students to compare the two methods.
- Demonstrate that the derivatives of functions such as ln x, $\sin^{-1} x$, and $\cos^{-1} x$ can be found using implicit differentiation. For example, given $y = \sin^{-1} x$, $\frac{dy}{dx}$ can be determined implicitly, as follows:
  - Since $y = \sin^{-1} x$, then $\sin y = x$
  - Thus $\cos y\frac{dy}{dx} = x$, which means $\frac{dy}{dx} = \frac{1}{\cos y}$
  - Since $\cos^2 y = 1 - \sin^2 y$, then $\cos y = \sqrt{1-\sin^2 y}$
  - Substituting $\sin y = x$, we get $\cos y = \sqrt{1-x^2}$.
  - Now substitute this into $\frac{dy}{dx} = \frac{1}{\cos y}$ to get $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
- Although full proof of the chain rule is uninformative, a graphing calculator can be used to show quite persuasively that for example $\frac{d}{dx}(\sin 3x)$ ought to be 3 cos 3x at any particular place x [given that $\frac{d}{dx}\sin x = \cos x$].

SUGGESTED ASSESSMENT STRATEGIES

As students develop confidence and proficiency computing the derivative of elementary functions, they are better prepared to solve more complex problems that involve derivative formulas. Assessment should focus on students' ability to recall derivatives of elementary functions and apply this knowledge appropriately when computing the derivative of a more complex function (using formulas, implicit differentiation, or logarithmic differentiation).

Observe

- As students are working on problems, circulate and provide feedback on their notation use. Have students verify their work using a graphing calculator.

Collect

- For a question where $f(x) = (x^2 + x)\ 3$ ask students to debate the effectiveness of finding the derivative using expansion, the chain rule, the product rule, and logarithmic differentiation. To what extent do their arguments illustrate their ability to expand on current mathematical idea?
- Have students present to the class research on some elementary functions and their derivatives.
- Discuss with students the merits of the various methods of taking derivatives. Have them summarize in writing their understanding of the merits of each. Work with them to develop a set of criteria to assess the summaries.

Self-Assessment

- To assess students' command of the chain rule, have them list common difficulties they encounter in taking derivatives. Let them work in pairs to develop checklists of reminders to use when they're checking their work.
- Ask students to summarize derivatives of elementary functions, making notes on notation errors. Allow students to complete assignments using their summaries.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 3 (Sections 3.3, 3.5, 3.6, 3.7, 3.8, 3.9)
  pp. 119, 151-154, 169
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 121, 125-126, 133-134, 137, 139, 143, 149, 158, 163, 168, 170
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 3 (Sections 3.1, 3.2, 3.4, 3.5, 3.6, 3.7 3.8)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 189, 191-196, 200, 204, 246, 257, 258, 261
- Calculus of a Single Variable, Sixth Edition
  Ch. 2 (Section 2.5)
  pp. 103, 105-107, 114-120,125-126, 315-316, 340, 380

PRESCRIBED LEARNING OUTCOMES

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

It is expected that students will:

- given the graph of $y = f(x)$:
  - graph $y = f'(x)$ and $y = f''(x)$
  - relate the sign of the derivative on an interval to whether the function is increasing or decreasing over that interval.
  - relate the sign of the second derivative to the concavity of a function
- determine the critical numbers and inflection points of a function
- determine the maximum and minimum values of a function and use the first and/or second derivative test(s) to justify their solutions
- use Newton's iterative formula (with technology) to find the solution of given equations, $f(x) = 0$
- use the tangent line approximation to estimate values of a function near a point and analyse the approximation using the second derivative

SUGGESTED INSTRUCTIONAL STRATEGIES

The first and second derivatives of a function provide a great deal of information concerning the graph of the function. This information is critical to solving problems involving calculus and helps students understand what the graph of a function represents.

- Ask students to compare the questions, "Where is $-x^3 + 14x^2 + 20x$ increasing most rapidly? Where is $-x^3 + 14x^2 + 20x$ increasing?" Have students develop similar questions.
- Have students work in teams of two to determine why the calculator is not helpful in questions such as: "let $f(x) = ax^3 - 5x^2$. Describe in terms of the parameter a, where $f(x)$ reaches a maximum or minimum."
- Have students use technology to explore features of functions such as the following:
  - when a function has maximum/minimum points or neither
  - where the inflection points occur
  - where curves are concave up or down
  - vertical or horizontal tangent lines
  - the relationship among graphs of the 1st, 2nd derivatives of a function and the function
  - the impact that endpoints have on the maximum/minimum
- Use a summary chart to demonstrate increasing and decreasing aspects of a function. Have students describe how this relates to the 1st derivative.
- Have groups of students learn particular concepts, (e.g., Newton's method) and teach these to their peers. Have them examine situations where it works very efficiently, situations where it works very slowly (e.g., $x^5 = 0$), and situations where it does not work (e.g., $x^{\frac{1}{3}} = 0$). Challenge them to develop hypotheses as to why it does or does not work.
- Have students research and report on the history of Newton's method for determining square roots (e.g., mentioning Heron of Alexandria, and the work of mathematicians in Babylon, India).
- Discuss with students the relative merits of using or not using a graphing calculator "to do" calculus. Present them with questions such as $g(x) = 8x^3 - 5x^2 + x - 3$ contrasted with $f(x) = \frac{1}{3}x^3 - 10{,}000x + 100$. Note that the calculator does not always give the necessary detail.

SUGGESTED ASSESSMENT STRATEGIES

Students demonstrate their understanding of first and second derivatives by relating the information they obtain from these to the graph of a function (i.e., critical numbers, inflection points, maximum and minimum values, and concavity).

Observe

- When reviewing students work, note the extent to which they can:
  - accurately determine the 1st and 2nd derivative
  - apply the respective tests
  - recognize and describe the relationship among the graphs of f(x), f′(x), f″(x)

Collect

- Ask the students to sketch a cubic graph and determine what the slopes of the tangent line would be at the local maximum and minimum points. Have them present their summary to the class. Consider the extent to which they can explain why the derivative is 0 at a local maximum/minimum.

Peer assessment

- Have students critique each others' graphs using criteria generated by the class. These criteria could include the extent to which:
  - a graph is appropriate to the function it is supposed to represent
  - the axes are accurately labelled
  - appropriate scales have been chosen for the axes
  - the graphs present smooth curves
  - domain, range, asymptotes, intercepts, and vertices have been correctly determined
  - inflection points, maximum and minimum points, and region of concavity have been correctly identified

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 4 (Sections 4.4, 4.6)
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 82, 127-128, 173, 247
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 3 (Section 3.10)
  Ch. 4 (Sections 41, 4.7)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 172, 270, 329
  Calculus of a Single Variable, Sixth Edition
  Ch. 2 (Section 2.6)
  Ch. 3 (Section 3.7)

PRESCRIBED LEARNING OUTCOMES

It is expected that students will solve applied problems from a variety of fields including the Physical and Biological Sciences, Economics, and Business.

It is expected that students will:

- solve problems involving displacement, velocity, acceleration
- solve related rates problems
- solve optimization problems (applied maximum/minimum problems)

SUGGESTED INSTRUCTIONAL STRATEGIES

Calculus was developed to solve problems that had previously been difficult or impossible to solve. Such problems include related rate and optimization problems, which arise in a variety of fields that students may be studying (e.g., the physical and biological sciences, economics, and business).

- Have a student demonstrate a "student trip" in front of the class — constant speed, accelerate, stop, slow down, stop, back up. Have students sketch a graph of displacement against time for the movements.
- Discuss average velocity over a specified interval, instantaneous velocity at specified times. Have students in pairs create a graph and have their partner perform the motion.
- Have students brainstorm examples of the need for calculus in the real world:
  - population growths of bacteria
  - the optimum shape of a container
  - water draining out of a tank
  - the path that requires the least time to travel
  - marginal cost and profit
- Challenge students to create their own "new" problems, which they must then try to solve. These problems could be used to develop tests or unit reviews.
- Have students use technology (e.g., graphing calculators) to investigate problems and confirm their analytical solutions graphically.
- Discuss with students the merits of using versus not using a graphing calculator "to do" calculus.

SUGGESTED ASSESSMENT STRATEGIES

Students develop their knowledge of derivatives by solving problems involving rates of change, maximum, and minimum. When assessing student performance in relation to these problems, it is important to consider students' abilities to make generalizations and predictions about how calculus is used in the real world.

Observe

- While students are working on problems involving derivatives, look for evidence they:
  - clearly understand the requirements of the problem
  - recognized when a strategy was not appropriate
  - can explain the process used to determine their answers
  - used a graphing calculator where appropriate to help visualize their solution
  - verified that their solutions where correct and reasonable
- Provide a number of problems where students are required to use average velocity **or** instantaneous velocity. Observe the extent to which students are able to:
  - determine whether the situation calls for the calculation of average velocity or instantaneous velocity
  - explain the differences between the two
  - provide other examples

Collect

- Assign a series of problems that require students to apply their knowledge of the dynamics of change: how, at certain time the speed of an object is related to its height, and how, at a certain time the speed of an object is related to the change in velocity.
- To discover how well students can recognize and explain the concepts of change and rapid change give them a problem such as:

  A bacterial colony grows at a rate $\frac{dy}{dt} = y(C - y)$. How large is the colony when it is growing most rapidly? In their analysis look for evidence that they understand the rate of change of "the rate of change."

Question

- While students are working on simple area problems based on real-life applications, ask them to explain the relationship between the graph's units (on each axis) and the units of area under the curve.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 4 (Sections 4.3, 5.4)
  pp. 97-98, 172, 180, 198, 202
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 182, 184, 195, 197, 209, 219-221, 247, 257
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 2 (Section 2.9)
  Ch. 3 (Section 3.11)
  Ch. 4 (Sections 4.2, 4.3, 4.9)

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 211, 290, 299, 363
- Calculus of a Single Variable, Sixth Edition
  Ch. 3 (Sections 3.2, 3.3, 3.8, 3.9)
  pp. 157, 171-175, 182-183

PRESCRIBED LEARNING OUTCOMES

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

It is expected that students will:

- explain the meaning of the phrase "F(x) is an antiderivative (or indefinite integral) of f(x)"
- use antiderivative notation appropriately (i.e., $\int f(x)dx$ for the antiderivative of $f(x)$)
- compute the antiderivatives of linear combinations of functions whose individual antiderivatives are known including:
  - $\int k\,dx = kx + C$
  - $\int x^r dx = \frac{x^{r+1}}{r+1} + C$ if $r \neq -1$
  - $\int \frac{dx}{x} = \ln|x| + C$
  - $\int e^x dx = e^x + C$
  - $\int \sin x\,dx = -\cos x + C$
  - $\int \cos x\,dx = \sin x + C$
  - $\int \sec^2 x\,dx = \tan x + C$
  - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
  - $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- compute $\int f(ax+b)\,dx$ if $\int f(u)\,du$ is known
- create integration formulas from the known differentiation formulas
- solve initial value problems using the concept that if $F'(x) = G'(x)$ on an interval, then $F(x)$ and $G(x)$ differ by a constant on that interval

SUGGESTED INSTRUCTIONAL STRATEGIES

Students need to recognize that antidifferentiation is the reverse of the differentiation process. This understanding will enable them to calculate or verify.

- Use a variety of methods to explain the need for the constant, "C":
  - computational: $\frac{dy}{dx}(C) = 0$
  - geometrical: if we draw curves y = F(x), y = F(x) + C, one curve is obtained from the other by simply lifting, so slopes of tangent lines match
  - kinematic: if v(t) (velocity) is specified, then $\int v(t)dt$ represents all possible position (displacement) functions (we can't reconstruct position from velocity unless we know where we were at some time)
- To ensure students understand the arbitrary nature of assigning letters to variables, use a reasonable variety of letters in doing integration problems. For example, $\int \sin 2u\,du$, $\int e^{at}dt$.
- Integrals such as $\int (1+2x)^9 dx$ or $\int 7e^{x/4} dx$ can be calculated by a "guess and check" process; there is no need at this stage for a substitution rule of integrals. For example, to find $\int 5\sin 6x\,dx$, a student might "guess" that an answer is cos 6x, check by differentiating to get -6 sin 6x, then adjust to determine that the indefinite integral is $-\frac{5}{6}\cos 6x + C$.
- Initial value problems can be treated very informally. For example, suppose that f′(x) = e^{2x} and f(1) = 3. What is f(x)? One antiderivative of e^{2x} is $\frac{e^{2x}}{2}$ (check by differentiating), but this function does not satisfy the initial condition. To fix it, students can simply remember that the general antiderivative is $\frac{e^{2x}}{2} + C$, then choose C appropriately.
- Play a game by challenging students to write down as fast as possible antiderivatives of simple functions, such as $\int$ f(au + b)du where $\int$ f(x)dx is one of the standard integrals obtained by reversing familiar differentiation results.
- Introduce students to integration tables or software applications for symbolic integration.

SUGGESTED ASSESSMENT STRATEGIES

In making connections between antidifferentiation and differentiation, students should relate the physical world to abstract calculus representations. To assess students' learning, it is appropriate to consider their oral and written explanations of the meaning of antidifferentiation as well as evidence from their computation, their drawing, and their problem-solving activities.

Peer Assessment

- Have students work in pairs to find antidifferentiation problems on the Internet (e.g., at university mathematics department web pages), try to solve the problems they find, and verify their solutions.

Observe/Question

- Ensure students' notation is correct at all times (e.g., $\int e^{3x} dx = \frac{e^{3x}}{3} + C$). It is not necessary to try to justify the dx part of the notation, but the C (constant) part should be understood. Use questions to check understanding, such as "Can you think of a function whose derivative is x^2?"
- Divide the students into small groups, and give them a problem whose conventional solution uses a technique that they have not learned, (e.g., general substitution or integration by parts). For example, ($\int xe^{3x}$ dx, or $\int x^2 \ln x \, dx$. Observe the degree of working control students have of antiderivatives (and derivatives) by seeing how efficiently they can experiment their way to an answer (i.e., generate a variety of plausible ideas).

Collect

- Collect samples of students' worksheets dealing with recovery of functions from their derivatives. Assess the extent to which students are able to compute the antiderivation of functions and determine the value of C, when given initial conditions.

Self/Peer Assessment

- Have students make up tests or quizzes on antidifferentiation techniques and exchange their tests with partners. The partners should check each others' work. Allow the partners time to discuss the work and help each other formulate a plan to address areas of weakness.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 6 (Sections 6.1, 6.2)
  pp. 190, 304 - 307
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 278, 279-282, 319, 325, 342, 347, 374, 490, 495
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 4 (Section 4.10)
  Ch. 5 (Section 5.4, 5.5)
  p. 585

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 382, 384, 388, 392, 581
  Calculus of a Single Variable, Sixth Edition
  pp. 241, 243, 246, 290-291, 366, 385

PRESCRIBED LEARNING OUTCOMES

It is expected that students will use antidifferentiation to solve a variety of problems.

It is expected that students will:

- use antidifferentiation to solve problems about motion of a particle along a line that involve:
  - computing the displacement given initial position and velocity as a function of time
  - computing velocity and/or displacement given suitable initial conditions and acceleration as a function of time
- use antidifferentiation to find the area under the curve $y = f(x)$, above the x-axis, from $x = a$ to $x = b$
- use differentiation to determine whether a given function or family of functions is a solution of a given differential equation
- use correct notation and form when writing the general and particular solution for differential equations
- model and solve exponential growth and decay problems using a differential equation of the form: $\frac{dy}{dt} = ky$
- model and solve problems involving Newton's Law of Cooling using a differential equation of the form: $\frac{dy}{dt} = ay + b$

SUGGESTED INSTRUCTIONAL STRATEGIES

Antidifferentiation enables students to solve problems that are otherwise unsolvable (e.g., problems related to exponential growth and decay). These problems are found in a variety of contexts, from science to business.

- Demonstrate how physics formulae about motion under constant acceleration can be justified using antidifferentiation. If a(t) = -g, then v(t) = -gt + C for some C, and therefore $s(t) = -\frac{1}{2}gt^2 + Ct + D$ for some D. The two constants of integration can be found by using initial conditions. Given g = 9.81 and a rock thrown upward at a given speed from a 100 m tower, it is possible to calculate the maximum height reached and the time it takes for the rock to hit the ground. There are many variations on this problem.
- To explain why antiderivatives can be used to solve area problems, let f(x) be some given function, and let A(u) be the area under y = f(x), above the x axis, from x=a to x=u. By looking at $\frac{A(u+h) - A(u)}{h}$, we can argue reasonably that A′(u) = f(u), at the same time reinforcing the concept of derivative from the definition. So A(u) is an antiderivative of f(u). A(a)=0.
- Many calculators have a numerical integration feature that approximates $\int_a^b f(x)dx$. Have students find the area from a to b under a curve y=f(x) for which they can find $\int$ f(x)dx. Compare this solution with the calculator's approximation.
- Give students a function such as f(x) = ln x and ask them to make a rough sketch of an antiderivative F(x) of f(x) such that F(1) = 0.
- Illustrate the wide applicability of the concepts by using examples that are not from the physical sciences. For example, let C(x) be the cost of producing x tons of a certain fertilizer. Suppose that (for reasonable x), C′(x) is about 30 - 0.02x. The cost of producing 2 tons is \$5000. What is the cost of producing 100 tons?
- Have students generate and maintain a list of phenomena other than radioactive decay that are described by the same differential equation. For example, the illumination I(x) that reaches x metres below the surface of the water can be described by $\frac{dI}{dx} = -kI$. Under constant inflation, the buying power V(t) of a dollar t years of now can be described by $\frac{dV}{dt} = -kV$.

SUGGESTED ASSESSMENT STRATEGIES

An understanding of antidifferentiation and its applications is essential to the solution of the "area under the curve" problems. Students can demonstrate their understanding of antidifferentiation ideas and skills through their problem-solving work.

- To check whether the meaning of the phrase "solution of a differential equation" is fully understood, have students work in groups to discuss and solve problems such as the following:
  - show that $y = x^8$ is a solution of the differential equation $x\frac{dy}{dx} = 8y$
  - find a solution of the above differential equation with y(2) = 16
- When students are asked to solve problems such as the following, verify that they are able to identify sin *kt* and cos *kt* as solutions of the differential equation and that they can find other solutions as well:
  - A weight hangs from an ideal spring. It is pulled down a few centimetres then released. If y is the displacement of the weight from its rest position, it turns out that $\frac{d^2y}{dt^2} = -k^2y$ for some constant k
- Take a problem (e.g., an exponential decay problem with several parts) and ask for the solution to be written up as a prose report, in complete sentences, with the reasons for each step clearly explained. Criteria for assessment include clarity and grammatical correctness.
- Pose the problem, "In how many different ways can we find the area under $y = x^2$, above the x axis, from $x = 0$ to $x = a$, exactly or approximately?" Students can write a report on this. Students should be able to identify
  - Archimedes' method
  - standard antiderivative method
  - approximation techniques of their own devising
- Ask students to use standard Internet resources to find information about differential equations in various areas of application, and to report on an area of interest to them. Assess the extent to which they are able to express their findings coherently and in their own words.

RECOMMENDED LEARNING RESOURCES

Print Materials

- Calculus: Graphical, Numerical, Algebraic
  Ch. 6 (Section 6.1, 6.4)
  pp. 262 - 263, 311, 333-334
- Calculus of a Single Variable Early Transcendental Functions, Second Edition
  pp. 278, 290, 373-374, 377, 381-382, 385, 409
- Single Variable Calculus Early Transcendentals, Fourth Edition
  Ch. 4 (Section 4.10)
  Ch. 5 (Sections 5.1, 5.4)
  pp. 586, 603, 611

Multimedia

- Calculus: A New Horizon, Sixth Edition
  pp. 328, 408, 433, 580, 601, 611
- Calculus of a Single Variable, Sixth Edition
  Ch. 4 (Section 4.4)
  Ch. 5 (Section 5.5)
  pp. 242, 362

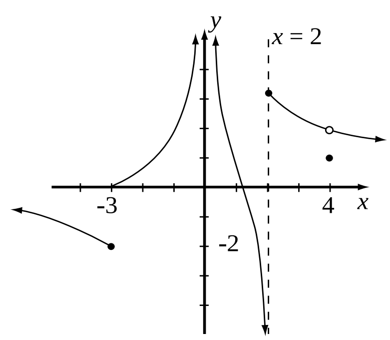
FUNCTIONS, GRAPHS AND LIMITS *(Functions and their Graphs)*

It is expected that students will represent and analyse inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions, using technology as appropriate.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will:	
• model and apply inverse trigonometric, base e exponential, natural logarithmic, elementary implicit, and composite functions to solve problems	c Find the area of the part of the first quadrant that is inside the circle $x^2+y^2=4$ and to the left of the line $x=u$. (The inverse sine function will be useful here.) c Suppose that under continuous compounding 1 dollar grows after t years to $(1.05)^t$ dollars. Find the number r such that $(1.05)^t=e^{rt}$. (This r is called the nominal yearly interest rate.)
• draw (using technology), sketch and analyse the graphs for rational, inverse trigonometric, base e exponential, natural logarithmic, elementary implicit and composite functions, for: - domain and range - intercepts	c Sketch the graph of $\frac{x^2}{x^2-1}$. c Write $\sin(\tan^{-1}x)$ in a form that does not involve any trigonometric functions. c a) Let $f(x)=\ln(ex)$. Sketch the graph of $f(x)$. b) Let $g(x)$ be the inverse function of $f(x)$. Sketch the graph of $g(x)$. c) Find an explicit formula for $g(x)$. c Sketch the curve $y=\ln(e^x+1)$.
• recognize the relationship between a base a exponential function (a>0) and the equivalent base e exponential function (convert $y=a^x$ to $y=e^{x(\ln a)}$) (Differentiation problems will also require students to demonstrate achievement of this outcome.)	c Let $f(x)=x^x$. a) Express $f(x)$ in the form $f(x)=e^{g(x)}$. b) Find the minimum value taken on by $f(x)$ on the interval $(0,\infty)$.
• determine using the appropriate method (analylitic or graaphing utility) the points where $f(x)=0$	c [No example for this prescribed learning outcome}

FUNCTIONS, GRAPHS AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • demonstrate an understanding of the concept of limit and notation used in expressing the limit of a function $f(x)$ as x approaches a: $\lim_{x \to a} f(x)$	c * Sketch and explain what each of the following means: a) $\lim_{x \to a} f(x) = L$ b) $\lim_{x \to a^-} f(x) = L$ c) $\lim_{x \to a^+} f(x) = L$ d) $\lim_{x \to a^+} f(x) = \infty$ e) $\lim_{x \to a^-} f(x) = \infty$ f) $\lim_{x \to \infty} f(x) = L$
• evaluate the limit of a function - analytically - graphically - numerically • distinguish between the limit of a function as x approaches a and the value of the function at $x = a$ • demonstrate an understanding of the concept of one-sided limits and evaluate one-sided limits • determine limits that result in infinity (infinite limits) • evaluate limits of functions as x approaches infinity (limits at infinity)	c Use a calculator to draw conclusions about: a) $\lim_{t \to 0} \frac{\sin t - t}{t^3}$ b) $\lim_{x \to 0^+} x^x$  c From the given graph find the following limits. If there is no limit, explain why: a) $\lim_{x \to 2^+} f(x)$ b) $\lim_{x \to 2^-} f(x)$ c) $\lim_{x \to -3^+} f(x)$ d) $\lim_{x \to -3^-} f(x)$ e) $\lim_{x \to -3} f(x)$

FUNCTIONS, GRAPHS AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

Prescribed Learning Outcomes	Illustrative Examples
	f) $\lim_{x\to 4} f(x)$ g) $f(4)$ h) $\lim_{x\to 0} f(x)$ C Find the limits: a) $\lim_{h\to 0}\frac{(1+h)^2-1}{h}$ b) $\lim_{t\to 6}\frac{17}{(t-6)^2}$ c) $\lim_{h\to 0}\frac{(2+h)^{-2}-\frac{1}{4}}{h}$ d) $\lim_{v\to 2}\frac{v^2+2v-8}{v^4-16}$ e) $\lim_{x\to 8^-}\frac{\lvert x-8\rvert}{x-8}$ f) $\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x}$ g) $\lim_{x\to 0}\frac{\sin 3x}{x}$ h) $\lim_{\theta\to 0}\frac{\tan 3\theta}{\theta}$ i) $\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 5\theta}$ j) $\lim_{x\to\infty}\frac{3x^2+5}{4-x^2}$ k) $\lim_{x\to\infty} x\sin\frac{1}{x}$ l) $\lim_{x\to\infty}\frac{\sqrt{x^2+4x}}{4x+1}$ m) $\lim_{x\to 0^+}\frac{1}{2+10^{\frac{1}{x}}}$ n) $\lim_{x\to 0^-}\frac{1}{2+10^{\frac{1}{x}}}$

FUNCTIONS, GRAPHS AND LIMITS *(Limits)*

It is expected that students will understand the concept of a limit of a function, notation used, and be able to evaluate the limit of a function.

Prescribed Learning Outcomes	Illustrative Examples
• determine vertical and horizontal asymptotes of a function, using limits	c Find the horizontal and vertical asymptotes of each of the following curves: a) $y=\dfrac{x}{x+4}$ b) $y=\dfrac{x^2}{x^2-1}$ c) $y=\dfrac{x^2}{x^2+1}$ d) $y=\dfrac{1}{(x-1)^2}$
• determine whether a function is continuous at $x=a$	c *Given $f(x)=\begin{cases}\sqrt{-x} & ; \quad x<0 \\ 3-x & ; \quad 0\le x\le 3 \\ (x-3)^2 & ; \quad x>3\end{cases}$ a) find: i) $\lim\limits_{x\to 0^+} f(x)$ ii) $\lim\limits_{x\to 0^-} f(x)$ iii) $\lim\limits_{x\to 0} f(x)$ iv) $\lim\limits_{x\to 3^-} f(x)$ v) $\lim\limits_{x\to 3^+} f(x)$ vi) $\lim\limits_{x\to 3} f(x)$ b) Where is the graph discontinuous. Why? c) Sketch the graph.

THE DERIVATIVE *(Concept and Interpretations)*

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • describe geometrically a secant line and a tangent line for the graph of a function at $x = a$. • define and evaluate the derivative at $x = a$ as: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ • define and calculate the derivative of a function using: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ or $f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ • use alternate notation interchangeably to express derivatives (i.e., $f'(x), \frac{dy}{dx}, y'$, etc.) • compute derivatives using the definition of derivative	c Calculate the derivatives of the given function directly from the definition: a) $f(x)=x^2+3x$ b) $g(x)=\frac{x}{x-1}$ c) $H(t)=\sqrt{t+1}$ c Using the definition of derivative, find the equation of the tangent line to the curve: $y=5+4x-x^2$ at the point $(2,9)$.
• distinguish between continuity and differentiability of a function at a point • determine when a function is non-differentiable, and explain why • determine the slope of a tangent line to a curve at a given point • determine the equation of the tangent line to a curve at a given point	c Determine where $f(x)=\|x-3\|$ is non differentiable. c Find an equation of the line tangent to the given curve at the point indicated. Sketch the curve: $y=\frac{1}{x^2+1}$ at $x=-1$ c *Find all the values of a for which the tangent line to $y=\ln x$ at $x=a$ is parallel to the tangent line to $y=\tan^{-1}x$ at $x=a$.

THE DERIVATIVE *(Concept and Interpretations)*

It is expected that students will understand the concept of a derivative and evaluate derivatives of a function using the definition of derivative.

Prescribed Learning Outcomes	Illustrative Examples
• for a displacement function $s = s(t)$, calculate the average velocity over a given time interval and the instantaneous velocity at a given time • distinguish between average and instantaneous rate of change	c The position of a particle is given by $s = t^3$. Find the velocity when $t = 2$. c *The displacement in metres of a particle moving in a straight line is given by $s = 5 + 4t - t^2$ where t is measured in seconds. a) Find the average velocity from $t = 2$ to $t = 3$, $t = 2$ to $t = 2.1$, and $t = 2$ to $t = 2.01$. b) Find the instantaneous velocity when $t = 2$. c) Draw the graph of s as a function of t and draw the secant lines whose slopes are average velocity as in a). d) Draw the tangent line whose slope is the instantaneous velocity in b).

THE DERIVATIVE *(Computing Derivatives)*

It is expected that students will determine derivatives of functions using a variety of techniques.

Prescribed Learning Outcomes	Illustrative Examples

It is expected that students will:

- compute and recall the derivatives of elementary functions including:
  - $\frac{d}{dx}(x^r) = rx^{r-1}$, r e $\Re$
  - $\frac{d}{dx}(e^x) = e^x$
  - $\frac{d}{dx}(\ln x) = \frac{1}{x}$
  - $\frac{d}{dx}(\cos x) = -\sin x$
  - $\frac{d}{dx}(\sin x) = \cos x$
  - $\frac{d}{dx}(\tan x) = \sec^2 x$
  - $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- use the following derivative formulas to compute derivatives for the corresponding types of functions:
  - constant times a function: $\frac{d}{dx}cu = c\frac{du}{dx}$
  - $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ (sum rule)
  - $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (product rule)
  - $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (quotient rule)
  - $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$ (power rule)
- use the Chain Rule to compute the derivative of a composite function:
  $\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$ or $\frac{d}{dx}(F(g(x))) = g'(x)F'(g(x))$

c Calculate the derivatives of the following functions:

a) $y = 3 - 4x - 5x^2 + 6x^2$

b) $z = \frac{S^5 - S^3}{15}$

c) $f(t) = \frac{4}{2-5t}$

d) $y = (2x+3)^6$

e) $y = (x^2+9)\sqrt{x^2+3}$

f) $f(x) = \sin 2x$

g) $y(x) = \cos^2(5-4x^3)$

h) $y = \ln(3t^2+6)$

i) $y = 2e^{-x}$

j) $y = \tan(e^x)$

k) $F(x) = \log_5(3x-8)$

l) $y = 2^{x^4}$

m) $y = \sin^{-1}(\sqrt{2x})$

n) $y = \tan^{-1}(3x)$

c Let $f(x) = \ln\left(x + \sqrt{4+x^2}\right)$. Find $f'(x)$ and simplify.

c A certain function $f(x)$ has $f(1) = 4$ and $f'(1) = 5$. Let $g(x) = \frac{1}{\sqrt{2f(x)+1}}$. Find $g'(1)$.

THE DERIVATIVE *(Computing Derivatives)*

It is expected that students will determine derivatives of functions using a variety of techniques.

Prescribed Learning Outcomes	Illustrative Examples
• compute the derivative of an implicit function.	c Find $\frac{dy}{dx}$ in terms of x and y if: a) $xy = x + 2y + 1$ b) $x^3y + xy^5 = 2$ c Find the equation of the tangent line at the given point: $x^2y^3 - x^3y^2 = 12$ at $(-1, 2)$ c Find y'' by implicit differentiation: $x^2 - y^2 = 1$ c The equation $x^4 + x^2y + y^4 = 27 - 6y$ defines y implicitly as a function of x near the point (2,1). Calculate y' and y'' at (2,1).
• use the technique of logarithmic differentiation.	c Let $y = (x+1)^2(2x+1)^3(3x+1)^4e^{5x}$. Use logarithmic differentiation to find $\frac{dy}{dx}$. c Using logarithmic differentiation, find a) y' if $y = x^x\ (x > 0)$ b) $\frac{dy}{dx}$ if $y = 5^{2x+1}$ c) $f'(x)$ if $f(x) = \frac{x^5\sqrt{3+x^2}}{(4+x^2)^3}$
• compute higher order derivatives	c Find y', y'', and y''' for the functions: a) $y = x^2 - \frac{1}{x}$ b) $y = \frac{x-1}{x+1}$ c Let $f(x) = \frac{1}{1+x}$. Find $f^{(7)}(x)$, the seventh derivative of $f(x)$.

APPLICATIONS OF DERIVATIVE *(Derivatives and the Graph of the Function)*

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • given the graph of $y = f(x)$: - graph $y = f'(x)$ and $y = f''(x)$ - relate the sign of the derivative on an interval to whether the function is increasing or decreasing over that interval. - relate the sign of the second derivative to the concavity of a function • determine the critical numbers and inflection points of a function. • determine the maximum and minimum values of a function and use the first and / or second derivative test(s) to justify their solutions	c For the following functions, find the critical numbers, the inflection points, and the vertical and horizontal asymptotes. Sketch the graph, and verify using a graphing calculator. a) $f(x) = \frac{x^3}{x^2 - 1}$ b) $y = \frac{3x^2}{2x^2 + 1}$ c) $f(x) = \frac{1}{x^2 - x}$ d) $f(x) = (x^2 - 1)^{\frac{2}{3}}$ c Find the critical and inflection points for $f(x) = -2xe^{-x}$ and sketch the graph of $f(x)$ for $x \geq 0$. c Sketch the graph $g(x) = x + \cos x$. Determine where the function is increasing most rapidly and least rapidly.
• use Newton's iterative formula (with technology) to find the solution of given equations, $f(x) = 0$	c Using a calculator, complete at least 5 iterations of Newton's method for $g(x) = x - \sin x + 1$ if $x_0 = -1$. c Explain why the function *f*, where $f(x) = \frac{2 \ln x}{1 + x^2}$, has exactly one critical number. Use Newton's Method to find that critical number, correct to two decimal places. c Use Newton's Method to find, correct to two decimal places, the *x*-coordinate of a point at which the curve $y = \tan x$ meets the curve $y = x + 2$. Also look at the problem by using the "solve" feature of your calculator and by graphing the curves.

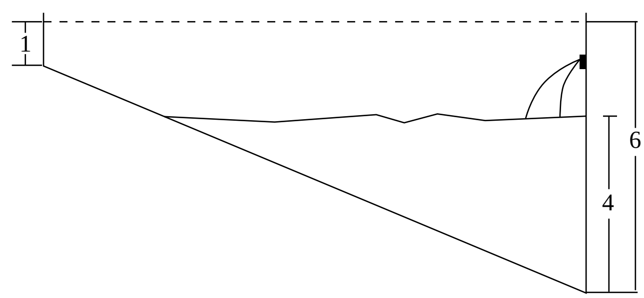
APPLICATIONS OF DERIVATIVE *(Derivatives and the Graph of the Function)*

It is expected that students will use the first and second derivatives to describe the characteristic of the graph of a function.

Prescribed Learning Outcomes	Illustrative Examples
• use the tangent line approximation to estimate values of a function near a point and analyse the approximation using the second derivative.	c Determine the tangent line approximation for $f(x) = \sin x$ in the neighbourhood of $x = 0$. Zoom in on both graphs and compare the results. c A certain curve has equation $e^{2y} + y = x^2$. Note that the point (1,0) lies on the curve. Use a suitable tangent line approximation to give an estimate of the y-coordinate of the point on the curve that has x-coordinate equal to 1.2. c A certain function $f(x)$ has derivative given by $f'(x) = \sqrt[3]{x^2 - 1}$. It is also known that $f(3) = 4$. a) Use the tangent line approximation to approximate $f(3.2)$. b) Use the second derivative of f to determine whether the approximation in part (a) is bigger or smaller than the true value of $f(3.2)$. c The diameter of a ball bearing is found to be 0.48 cm, with possible error ± 0.005 cm. Use the tangent line approximation to approximate: a) the largest possible error in computing the volume of the ball bearing, b) the maximum percentage error

APPLICATIONS OF DERIVATIVE *(Applied Problems)*

It is expected that students will solve applied problems from a variety of fields including the physical and biological sciences, economics, and business.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • solve problems involving distance, velocity, acceleration	c The average velocity of a particle in the time interval from 1 to t is $\frac{e^t}{e^t - 1}$ for $t > 1$. Find the displacement and velocity at time $t = 2$.
• solve related rate problems (rates of change of two or more related variables that are changing with respect to time)	c Car P (a police car) was travelling in a northerly direction along the y-axis at a steady 50 kilometres per hour, while car Q was travelling eastward along the x-axis at varying speeds. At the instant when P was 60 metres north of the origin, Q was 90 metrs east of the origin. A radar unit on P recorded that the straight-line distance between P and Q was, at that instant, increasing at the rate of 80 kilometres per hour. How fast was car Q going? c *A swimming pool is 25 metres wide and 25 metres long. When the pool is full, the water at the shallow end is 1 metre deep, and the water at the other end is 6 metres deep. The bottom of the pool slopes at a constant angle from the shallow end to the deep end. Water is flowing into the deep end of the pool at 1 cubic meter per minute. How fast is the depth of water at the deep end increasing when the water there is 4 metres deep?  c *Boyle's Law* states that if gas is compressed but kept at constant temperature, then the pressure P and the volume V of the gas are related by the equation $PV=C$, where C is a constant that depends on temperature and the quantity of gas present. At a certain instant, the volume of gas inside a pressure chamber is 2000 cubic centimetres, the pressure is 100 kilopascals, and the pressure is increasing at 15 kilopascals per minute. How fast is the volume of the gas decreasing at this instant?

APPLICATIONS OF DERIVATIVE *(Applied Problems)*

It is expected that students will solve applied problems from a variety of fields including the physical and biological sciences, economics, and business.

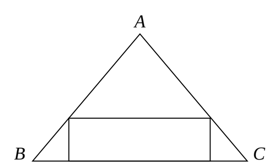
Prescribed Learning Outcomes	Illustrative Examples

Prescribed Learning Outcomes

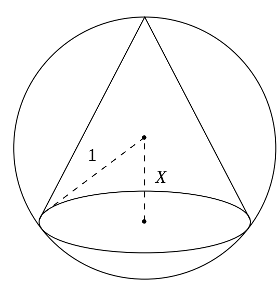
- solve optimization problems (applied maximum/minimum problems)

Illustrative Examples

- A striped ball is thrown vertically upward, having a velocity of 12 m/s. After t seconds its altitude is represented by $s = 12t - 4.9t^2$. At what instant will it reach a maximum height and how high will it rise?
- *Construct an open-topped box with a maximum volume from a 20 cm x 20 cm piece of cardboard by cutting out four equal corners. Use more than one method. Compare your results.
- Triangle ABC has AB=AC=8, BC=10. Find the dimensions of the rectangle of maximum area that can be inscribed in ΔABC with one side on the rectangle on BC.



- A pharmaceuticals company can sell its veterinary antibiotic for a price of $450 per unit. Assume that the total cost of producing x units in a year is given by $C(x)$ where $C(x) = 800\ 000 + 50x + 0.004x^2$. How many units per year should the company produce in order to maximize yearly profit?
- Find the volume of the cone of maximum volume that can be inscribed in a sphere of radius 1. Hint: let X be as shown in the diagram.



ANTIDIFFERENTIATION *(Recovering Functions from their Derivatives)*

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will: • explain the meaning of the phrase "$F(x)$ is an antiderivative (or indefinite integral) of $f(x)$" • use antiderivative notation appropriately (i.e., $\int f(x)dx$ for the antiderivative of $f(x)$) • compute the antiderivatives of linear combinations of functions whose individual antiderivatives are known including: - $\int k\,dx = kx + C$ - $\int x^r dx = \frac{x^{r+1}}{r+1} + C$ if $r \neq -1$ - $\int \frac{dx}{x} = \ln\lvert x\rvert + C$ - $\int e^x dx = e^x + C$ - $\int \sin x\,dx = -\cos x + C$ - $\int \cos x\,dx = \sin x + C$ - $\int \sec^2 x\,dx = \tan x + C$ - $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$ - $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$ • compute $\int f(ax+b)\,dx$ if $\int f(u)\,du$ is known	c Is it true that $\int 2xe^{2x}dx = xe^{2x} - \frac{e^{2x}}{2} + C$? c Is it true that $\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$? c Is it true that $\int x\cos x\,dx = \frac{x^2}{2}\sin x + C$? c Evaluate the following indefinite integrals: a) $\int (1+3x)^5 dx$ b) $\int e^{x/2}\,dx$ c) $\int \frac{5dx}{(1+4x)^2}$ c Suppose that $\frac{d^2y}{dx^2} = \sin \pi x$. a) Find a general formula for $\frac{dy}{dx}$ as a function of x. b) Find a general formula for y as a function of x.
• create integration formulas from the known differentiation formulas	c Let a be a positive constant other than 1. Find $\int a^x dx$. c Use the fact that $\sec^2 x = 1 + \tan^2 x$ to find $\int \tan^2 x\,dx$. c Use the identity $\cos 2x = 1 - 2\sin^2 x$ to find $\int \sin^2 x\,dx$. c Find $\int x\cos 3x\,dx$ by guessing that one antiderivative is $x\sin 3x$, differentiating, and then adjusting your guess.

ANTIDIFFERENTIATION *(Recovering Functions from their Derivatives)*

It is expected that students will recognize antidifferentiation (indefinite integral) as the reverse of the differentiation process.

Prescribed Learning Outcomes	Illustrative Examples
• solve initial value problems using the concept that if $F'(x) = G'(x)$ on an interval, then $F(x)$ and $G(x)$ differ by a constant on that interval	c Suspose that $f''(t) = 3t^2$ for all t and $f(1) = 2$, $f'(1) = 5$. Find a formula for $f(t)$. c It is known that $f'(x) = e^{x/2}$ and $f(6 \ln 2) = 10$. Find a general formula for $f(x)$. c a) Verify that the function f given by $f(x) = 3e^{-x} \sin 3x - e^{-x} \cos 3x$ is an antiderivative of $10e^{-x} \cos 3x$. b) Use the result of part a) to find an antiderivative $G(x)$ of $10e^{-x} \cos 3x$ such that $G(0) = 5$.

ANTIDIFFERENTIATION *(Applications of Antidifferentiation)*

It is expected that students will use antidifferentiation to solve a variety of problems.

Prescribed Learning Outcomes	Illustrative Examples
It is expected that students will:	
• use antidifferentiation to solve problems about motion along a line that involve: - computing the displacement given initial position and velocity as a function of time - computing velocity and/or displacement given suitable initial conditions and acceleration as a function of time	c A particle is moving back and forth along the x-axis, with velocity at time t given by $v(t) = \sin(t/2)$. When $t = 0$, the particle is at the point with coordinates (4,0). Where is the particle at time $t = \pi$? c Let a be a given positive number. Find the area of the region which is under the curve with equation $y = \frac{1}{x}$, above the x-axis, and between the vertical lines $x = a$ and $x = 2a$. Simplify your answer.
• use antidifferentiation to find the area under the curve $y = f(x)$, above the x-axis, from $x = a$ to $x = b$	c Use calculus to find the area under the curve, above the x-axis, and lying between the lines $x = \frac{1}{\sqrt{3}}$ and $x = \sqrt{3}$.
• use differentiation to determine whether a given function or family of functions is a solution of a given differential equation • use correct notation and form when writing the general and particular solution for differential equations • model and solve exponential growth and decay problems using a differential equation of the form: $\frac{dy}{dt} = ky$	c a) Verify that $y = \sin 3t$ is a solution of the differential equation $y'' = -9y$. b) Find a solution of the above differential equation that is not a constant multiple of $\sin 3t$. c) Find a solution y of the differential equation such that $y(0) = 2$ and $y'(0) = 1$. c *Torricelli's Law says that if a tank has liquid in it to a depth h, and there is a hole in the bottom of the tank, then liquid leaves the tank with a speed of $\sqrt{2gh}$. Use the metre as the unit of length and measure time in seconds. Then g is about 9.81. Take in particular a cylindrical tank with base radius R, with a hole of radius r at the bottom. a) Verify that h satisfies the differential equation $\frac{dh}{dt} = \frac{r^2}{R^2}\sqrt{2gh}$ b) Verify that for any constant C, the function $h(t)$ is a solution of the above equation, where $h(t) = \frac{1}{4}\left(C + \frac{\sqrt{2g}\,r^2}{R^2}t^2\right)^2$ c) A hot water tank has a base radius of 0.3 metres and is being drained through a circular hole of radius 0.1 at the bottom of the tank. At a certain time, the depth of the water in the tank is 1.5 metres. What is the depth of the water 30 seconds later?

ANTIDIFFERENTIATION *(Applications of Antidifferentiation)*

It is expected that students will use antidifferentiation to solve a variety of problems.

Prescribed Learning Outcomes	Illustrative Examples
	c *Shortly after a person was exposed to iodine-131, the level of radioactivity in the person's thyroid was 20 times the "safe" level. Three hours later, the level had decreased to 15.5 time the safe level. How much longer must the person wait until the level of radioactivity in the thyroid reaches "safe" territory? c *A simple model for the growth of a fish of a particular species goes as follows: Let M be the maximum length that can be reached by a member of that species. At the instant when the length of an individual of that species is y, the length of the individual is growing at a rate proportional to $M-y$, that is, y satisfies the differential equation $\frac{dy}{dt}=k(M-y)$ for some constant k. a) Let $w = M-y$. Verify that w satisfies the differential equation $\frac{dw}{dt}=-kw$. b) Write down the general solution of the differential equation of part a). Then write down a general formula for y as a function of time. c) Suppose that for a certain species, the maximum achievable length M is 60 cm, and that when t is measured in years, k = 0.05. A fish of that species has current length of 10 cm. How long will it take to reach a length of 20c m?
• model and solve problems involving Newton's Law of Cooling using a differential equation of the form: $\frac{dy}{dt}=ay+b$	c *Coffee in a well-insulated cup started out at 95°, and was brought into an office that was held at a constant 20°. After 10 minutes, the temperature of the coffee was 90°. Assume that Newton's Law of Cooling applies. How much additional time must elapse until the coffee reaches 80°?